An alternative ending to "Pleasant extensions retaining algebraic structure"

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Abstract

The culmination of the two recent papers [4, 5] was a proof of the norm convergence in $L^2(\mu)$ of the quadratic nonconventional ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N}(f_1\circ T_1^{n^2})(f_2\circ T_1^{n^2}T_2^n)\qquad f_1,f_2\in L^\infty(\mu)$$

associated to an arbitrary probability-preserving $\mathbb{Z}^2$-system (X,μ,T_1,T_2). This is a special case of the Bergelson-Leibman conjecture on the norm convergence of polynomial nonconventional ergodic averages [7].

That proof relied on some new machinery for extending probability-preserving $\mathbb{Z}^d$-systems to obtain simplified asymptotic behaviour for various nonconventional averages such as the above. The engine of this machinery is formed by some detailed structure theorems for the 'characteristic factors' that are available for some such averages after ascending to a suitably-extended system. However, these new structure theorems underwent two distinct phases of development, separated by the discovery of some new technical results in Moore's cohomology theory for locally compact groups [1]. That discovery enabled a significant improvement to the main structure theorem (Theorem 1.1 in [4]), which in turn afforded a much shortened proof of convergence. However, since the proof of convergence using the original structure theorem required some quite different ideas that are now absent from [4, 5], I have recorded it here in case it has some independent interest.

Contents

1 Introduction

This note records a proof of a new instance of the Bergelson-Leibman Conjecture on norm convergence of polynomial nonconventional ergodic averages:

Theorem 1.1. *If $T_1, T_2 : \mathbb{Z} \curvearrowright (X, \mu)$ are commuting invertible probability-preserving transformations of a standard Borel probability space then the averages*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T_1^{n^2})(f_2 \circ T_1^{n^2}T_2^n)$$

converge in $L^2(\mu)$ as $N \to \infty$ for any $f_1, f_2 \in L^\infty(\mu)$.

The proof of the present paper has been superseded by an improved approach in [4, 5], enabled by a recent development in the cohomology of compact groups ([1]). Originally, the proof recorded below was contained in a Part III to the sequence [4, 5], and I have maintained a presentation of it here in case it has any independent interest.

The proof of Theorem 1.1 follows a strategy that has emerged by stages in work of Furstenberg [12], Conze and Lesigne [9, 10, 11], Furstenberg and Weiss [13], Host and Kra [16, 17], Ziegler [28] and a number of others, and in the papers [6, 2, 4, 5] (see the introduction to [4] for a more complete history). We seek an extension of an initially-given system (X, μ, T_1, T_2), say $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2) \to (X, \mu, T_1, T_2)$, such that for the extended system the analogous nonconventional averages admit a 'simple' pair of factors $\xi_i : (\tilde{X}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2) \to (Y_i, \nu_i, S_{i,1}, S_{i,2})$ that is **characteristic**, in that

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ \tilde{T}_1^{n^2})(f_2 \circ \tilde{T}_1^{n^2}\tilde{T}_2^n) \sim \frac{1}{N}\sum_{n=1}^{N}(\mathsf{E}_{\tilde{\mu}}(f_1\,|\,\xi_1) \circ \tilde{T}_1^{n^2})(\mathsf{E}_{\tilde{\mu}}(f_2\,|\,\xi_2) \circ \tilde{T}_1^{n^2}\tilde{T}_2^n)$$

as $N \to \infty$ for all $f_1, f_2 \in L^\infty(\tilde{\mu})$, where we write $f_N \sim g_N$ to denote that $\|f_N - g_N\|_2 \to 0$ as $N \to \infty$.

These factors reduce our problem to proving convergence in case each f_i is ξ_i-measurable. Informally we refer to an extension that admits useful characteristic factors for some averages as a **pleasant** extension for those averages. The construction of a pleasant extension in this paper will rely on some of the results from [4, 5] (or, more precisely, from the incarnations of [4, 5] from before the above-mentioned re-write). In [4] we set up some general machinery for constructing extensions of abstract probability-preserving systems, showing in particular how to obtain the useful property of satedness with respect to an idempotent class of systems. In [5] we brought this machinery to bear on the problem of obtaining pleasant extensions for the linear nonconventional averages

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T^{n\mathbf{p}_1})(f_2 \circ T^{n\mathbf{p}_2})(f_3 \circ T^{n\mathbf{p}_3})$$

associated to a system $T : \mathbb{Z}^2 \curvearrowright (X, \mu)$ and a triple of distinct directions $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3 \in \mathbb{Z}^2$ that lie in general position with the origin. The difficulty of that construction results from the insistence that the pleasant extension should retain the algebraic relations among the transformations $T^{\mathbf{p}_i}$ that must follow from the linear dependence of the $\mathbf{p}_i$. In the previous version of [5] we eventually obtained a description of some characteristic factors for these linear averages that can be secured in an extended system as joins of various isotropy factors and a two-step distal $\mathbb{Z}^2$-system with compact Abelian fibres of a special kind called a 'directional CL-system'.

Theorem 1.2 (Pleasant extensions for general-position triple linear averages)**.** *For each $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{Z}^2$ that are in general position with the origin, let (m_i, m_{ij}, m_{ik}) be a relatively prime triple of nonzero integers such that $m_i\mathbf{p}_i + m_{ij}(\mathbf{p}_i - \mathbf{p}_j) + m_{ik}(\mathbf{p}_i - \mathbf{p}_k) = \mathbf{0}$. Then any system $T : \mathbb{Z}^2 \curvearrowright (X, \mu)$ has an extension $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}) \to (X, \mu, T)$ in which for every choice of such $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ the averages*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ \tilde{T}^{n\mathbf{p}_1})(f_2 \circ \tilde{T}^{n\mathbf{p}_2})(f_3 \circ \tilde{T}^{n\mathbf{p}_3}), \qquad f_1, f_2, f_3 \in L^\infty(\tilde{\mu}),$$

admit a characteristic triple of factors $\tilde{\xi}_i$, $i = 1, 2, 3$, of the form

$$\tilde{\xi}_i = \zeta_0^{\tilde{T}^{\mathbf{p}_i}} \vee \zeta_0^{\tilde{T}^{\mathbf{p}_i} = \tilde{T}^{\mathbf{p}_j}} \vee \zeta_0^{\tilde{T}^{\mathbf{p}_i} = \tilde{T}^{\mathbf{p}_k}} \vee \eta_i$$

where the target of η_i is a $(\mathbf{p}_i, m_{ij}(\mathbf{p}_i - \mathbf{p}_j), m_{ik}(\mathbf{p}_i - \mathbf{p}_k))$-directional CL-system (so certainly a two-step Abelian system) when $\{i, j, k\} = \{1, 2, 3\}$.

The definition of directional CL-systems will be given in Subsection 3.1 below. The above theorem no longer appears in [5], because it was subsequently discovered that a cohomological argument using the new continuity results for Moore cohomology in [1] enabled an arbitrary directional CL-system to be factorized into further isotropy factors and a two-step pro-nilsystem. This leads to an improved version of the above theorem in which η_i may itself simply be taken to be a two-step $\mathbb{Z}^2$-pro-nilsystem, and this improvement in turn leads to a much-shortened proof of convergence. The improved structural result now appears as Theorem 1.1 in [5], and the new proof of convergence is given in Section 5 of that paper. However, the theorem above can still be quite quickly deduced from the arguments that appear in [5]: Lemma 4.35 of [5] provides solutions to the 'directional CL-equations', and given this a fairly simple modification of the arguments from the current Subsection 4.6 of [5] yields a proof of the above structure theorem in place of its newer improvement.

The purpose of the present note is to retain a record of the proof of Theorem 1.1 using Theorem 1.2 above. Since Theorem 1.2 still lies within easy reach from the new contents of [5], I will simply assume it here.

A top-level outline of the older proof proceeds as follows. From a careful study of the possible joinings among directional CL-systems, we will be able to obtain a rather stronger characteristic-factor result for our advertised nonconventional quadratic averages. In part, the extra strength of this result will derive from a reduction to considering $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-systems for a pair of directions $\mathbf{n}_2$, $\mathbf{n}_3 \in \mathbb{Z}^2$ and a finite-index subgroup $\Gamma \leq \mathbb{Z}^2$, rather than $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-systems for a single direction $\mathbf{n}_1$.

In terms of these systems our pleasant extensions for our polynomial averages are as follows.

Theorem 1.3 (Pleasant extensions for quadratic averages)**.** *Any ergodic system of two commuting transformations $T_1, T_2 \curvearrowright (X, \mu)$ has an ergodic extension $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2) \to (X, \mu, T_1, T_2)$ in which the averages*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T_1^{n^2})(f_2 \circ T_1^{n^2}T_2^{n})$$

admit characteristic factors of the form

$$\xi_1 = \xi_2 := \bigvee_{m\geq 1} \zeta_0^{T_1^m} \vee \zeta_0^{T_2} \vee \bigvee_{h\geq 1} \eta_h,$$

where η_h is a factor of $(\tilde{X}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2)$ whose target is a $(h\mathbb{Z}^2, (h, 0), (0, h))$-directional CL-system for the finite-index sublattice $h\mathbb{Z}^2 := \{(hm, hn) : m, n \in \mathbb{Z}\}$.

We will bring Theorem 1.2 to bear on proving Theorem 1.3 via the well-known van der Corput estimate. Note that, unlike Theorem 1.2, we will prove Theorem 1.3 only for ergodic systems, and obtain ergodic extensions as a result. In fact the proof we give works equally well without this additional requirement, but the version formulated above will be more convenient for our proof of convergence.

After proving Theorem 1.3, we proceed towards the proof of Theorem 1.1 through a careful analysis of how functions measurable with respect to the factor $\xi_1 = \xi_2$ above behave upon composition with powers of T_1 and T_2. Although our methods for controlling the images of functions upon iterating an $(h\mathbb{Z}^2, (h,0), (0,h))$-directional CL-system are rather clumsy, we will find that the simplification afforded by Theorem 1.3 is still enough to enable a more-or-less direct proof of Theorem 1.1. In the present note this relies on adapting a strategy developed by Host and Kra in [16] for the treatment of the triple linear averages $\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T^n)(f_2 \circ T^{2n})(f_3 \circ T^{3n})$ for a single transformation T.

Notational remark In this note we will make free use of notations and definitions introduced in [4] and [5]. $\lhd$

2 A cohomological proposition

In the later stages of Section 3 below we will make crucial use of a technical proposition allowing us to re-write certain cocycles in a very explicit form. It will enable a final, extremely concrete re-writing of the quadratic nonconventional averages so that they are susceptible to a more direct analysis. We prove the needed technical result in this section as Proposition 2.1, preferring to separate it from the main steps in the proof of Theorem 1.1. Surprisingly, this will rest on a continuity result for certain measurable cohomology groups under taking inverse limits of the base groups, which will apply after we suitably re-cast the data we wish to simplify[1]. We will therefore need to call on the measurable cohomology theory for compact Abelian groups, as developed by Moore in his important sequence of papers [22, 23, 24]. We recall or prove those cohomological facts that we need in Appendix A.

Remark on notation We will write $\{\cdot\} : \mathrm{S}^1 \to [0,1)$ for the inverse to the bijection $\theta \mapsto \mathrm{e}^{2\pi\mathrm{i}\theta}$ and $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ for the usual 'integer part' function, so these maps are related by the equation $s - \lfloor s \rfloor = \{\mathrm{e}^{2\pi\mathrm{i}s}\}$, both sides of which give the

[1]This continuity result is a precursor from Moore's original papers of the more recent results of [1].

usual 'fractional part' of $s \in \mathbb{R}$. $\triangleleft$

Proposition 2.1 (Processing certain individual coboundary equations)**.** *Suppose that U is a compact metrizable Abelian group and $\psi : \mathbb{Z}^2 \to U$ is a homomorphism such that $\overline{\psi(\mathbb{Z}\mathbf{e}_1)} \cap \overline{\psi(\mathbb{Z}\mathbf{e}_2)} = \{0\}$ and $\overline{\psi(\mathbb{Z}\mathbf{e}_1)} \cdot \overline{\psi(\mathbb{Z}\mathbf{e}_2)}$ has finite index in U, and that $\sigma : \mathbb{Z}^2 \times U \to \mathrm{S}^1$ is a cocycle over the corresponding rotation action R_ψ of $\mathbb{Z}^2$ on (U, m_U). Suppose in addition that for each $i = 1, 2$ there are Borel maps $b_i : U \to \mathrm{S}^1$ and $c_i : U \to \mathrm{S}^1$ so that c_i is $R_{\psi(\mathbf{e}_i)}$-invariant and*

$$\sigma(\mathbf{e}_i, \,\cdot\,) = \Delta_{\psi(\mathbf{e}_i)} b_i \cdot c_i.$$

Then there are Borel maps $b'_i : U \to \mathrm{S}^1$ such that each $c'_i := c_i \cdot \Delta_{\psi(\mathbf{e}_i)} b'_i : U \to \mathrm{S}^1$ is a map of the form

$$c'_i(u) = \alpha_i(u) \cdot \exp\Big(2\pi\mathrm{i} \sum_{j=1}^{J_i} a_{i,j}(u) \{\chi_{i,j}(\psi(\mathbf{e}_i))\}\{\gamma_{i,j}(u)\}\Big)$$

for some function $\alpha_i : U \to \mathrm{S}^1$ that factorizes through a finite *quotient group of U, functions $a_{i,j} : U \to \mathbb{Z}$ for $j = 1, 2, \ldots, J_i$ that also factorize through this finite quotient group of U, and characters $\chi_{i,1}, \chi_{i,2}, \ldots, \chi_{i,J_i} \in \widehat{U}$ and $\gamma_{i,1}, \gamma_{i,2}, \ldots, \alpha_{i,J_i} \in \overline{\psi(\mathbb{Z}\mathbf{e}_i)}^\perp$. Therefore we can write instead*

$$\sigma(\mathbf{e}_i, \,\cdot\,) = \Delta_{\psi(\mathbf{e}_i)}(b_i \cdot b'_i) \cdot c'_i$$

with c'_i a map of this special form.

Remarks 1 Simply by playing around with examples of functions c_i that are already of the special form appearing above, it is not hard to show that there are quite nontrivial examples of $\mathbb{Z}^2$-systems admitting cocycles that satisfy the conditions of this proposition. For instance, let $w_1, w_2 \in \mathrm{S}^1$ be transcendental and algebraically independent over $\mathbb{Q}$ when identified with classes in $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and also such that $0 < \{w_1\}, \{w_2\} < 1/50$ and observe that if we define $\theta' \in \mathrm{S}^1$ by $\{\theta'\} = \{w_1\}\{w_2\}$ then the Borel map

$$c_1 : (\mathrm{S}^1)^2 \to \mathrm{S}^1 : (t_2, z_2) \mapsto z_2 \cdot \exp(-2\pi\mathrm{i}\{w_1\}\{t_2\}) = \exp(2\pi\mathrm{i}(\{z_2\} - \{w_1\}\{t_2\}))$$

satisfies

$$\Delta_{(w_2,\theta')} c_1(t_2, z_2) = -\lfloor \{w_2\} + \{t_2\} \rfloor \cdot w_1.$$

Now let $U := \mathrm{S}^1 \times (\mathrm{S}^1)^2$ and $\psi : (m, n) \mapsto (mw_1, n(w_2, \theta'))$ (this has dense image by the algebraic independence of $\{w_1\}$, $\{w_2\}$ and 1 over $\mathbb{Q}$), and define

$$\sigma_1(\mathbf{e}_1, (t_1, t_2, z_2)) := c_1(t_2, z_2)$$

and

$$\sigma_2(\mathbf{e}_2,(t_1,t_2,z_2)) := -\lfloor\{t_2\}+\{w_2\}\rfloor\cdot t_1 = \Delta_{(0,w_2,\theta')}b(t_1,t_2,z_2)\cdot c_2(t_1)$$

where

$$b(t_1,t_2,z_2) = \exp(2\pi\mathrm{i}\{t_1\}\{t_2\}) \qquad \text{and} \qquad c_2(t_1) = \exp(-2\pi\mathrm{i}\{t_1\}\{w_2\}).$$

We can now check immediately that $\Delta_{w_1}\sigma(\mathbf{e}_2,\,\cdot\,) = \Delta_{(w_2,\theta')}\sigma(\mathbf{e}_1,\,\cdot\,)$, so this does indeed define a cocycle over the rotation action R_ψ that admits functions b, c_1 and c_2 as in the above proposition. Furthermore, since $c_2(t_1)$ is $R_{(w_2,\theta')}$-invariant and takes continuum-many different values, it cannot be an $R_{(w_2,\theta')}$-quasi-coboundary (since for this to be true its values would be restricted to the eigenvalue group of some rotation on a compact metrizable Abelian group, and such an eigenvalue group would be countable); thus, in a sense, this c_2 does not admit further simplification in any obvious way, and similar remarks apply to c_1.

The importance of the above proposition is that it tells us that all such examples must be 'finite-dimensional' up to cohomology, and as the proof will show the reason behind this is very much a cohomological one (in particular, it will rest on the continuity of $\mathrm{H}^2(\cdot,\cdot)$ under inverse limits in the first argument, recalled below). Although such a result seems quite surprising a priori, we note that it does have a precedent in the study of pro-nilsystems as characteristic factors, where it is shown that towers of Abelian isometric extensions that are initially characterized by the Conze-Lesigne equation and its higher-step analogs can always be represented as inverse limits of finite-dimensional examples (see, in particular, [25, 17, 28]).

2 It seems likely that a version of this result is available without the simplifying assumption that $\overline{\psi(\mathbb{Z}\mathbf{e}_1)}\cap\overline{\psi(\mathbb{Z}\mathbf{e}_2)} = \{0\}$, but we make it here as this is the only case we will need and this assumption does lead to a much lighter presentation. $\lhd$

Proof Let $w_i := \psi(\mathbf{e}_i)$ and $K_i := \overline{\psi(\mathbb{Z}\mathbf{e}_i)}$ for $i = 1, 2$. We will make use of the cocycle condition

$$\Delta_{w_1}\sigma(\mathbf{e}_2,\,\cdot\,) = \Delta_{w_2}\sigma(\mathbf{e}_1,\,\cdot\,).$$

First, because there are only finitely many cosets of $K_1\cdot K_2$ in U and these are preserved by both of the rotations R_{w_1} and R_{w_2}, the desired conclusion clearly follows overall if we prove it separately within each of these cosets, and so we now simply assume that $U = K_1\cdot K_2$. Given this, the condition that $K_1\cap K_2 = \{0\}$ means we may assume $U = K_1\times K_2$ and correspondingly denote points of U as ordered pairs (u_1,u_2) in this product group.

Next, by adjusting the whole of σ by $\Delta_\psi b_1$, we may assume simply that $b_1 \equiv 1$. Given this, now substituting our expressions for $\sigma(\mathbf{e}_i,\,\cdot\,)$ into the commutativity condition we obtain

$$\Delta_{w_1}(\Delta_{w_2} b_2 \cdot c_2) = \Delta_{w_2} c_1.$$

We will deduce our desired conclusion from this equation in several small steps.

Step 1 We first focus our attention on the map b_2, with the goal of proving that it admits a factorization as

$$b_2(u_1,u_2) = \alpha(u_1,u_2)\cdot\rho_1(u_1,u_2)\cdot\rho_2(u_1,u_2)\cdot b_2'(u_1,u_2),$$

where $\alpha : K_1\times K_2 \to \mathrm{S}^1$ factorizes through some finite quotient group of $K_1\times K_2$, ρ_1 has the property that that $\rho_1(u_1,\,\cdot\,)$ is a member of $\mathcal{E}(K_2)$ for Haar-almost every $u_1 \in K_2$, ρ_2 has the symmetric property and b_2' is of the form

$$b_2'(u_1,u_2) = \exp\left(2\pi\mathrm{i}\sum_{j=1}^{J}\{\gamma_j(u_1)\}\{\chi_j(u_2)\}\right)$$

for some $\gamma_1, \gamma_2, \ldots, \gamma_J \in \widehat{K_1}$ and $\chi_1, \chi_2, \ldots, \chi_J \in \widehat{K_2}$. This will occupy the first five steps (the bulk of the proof).

Our first step amounts to a simple re-interpretation of the various data in hand. Observe that the right-hand side of the commutativity equation above is R_{w_1}-invariant, while the left-hand side is an R_{w_1}-coboundary. This implies that

- $\Delta_{w_2} c_1$ takes values in $\widehat{K_1}(w_1)$, and
- for almost every $u_2 \in K_2$ the map $\Delta_{w_2} b_2(\,\cdot\,,u_2)\cdot c_2(\,\cdot\,)$ is an eigenfunction on the subgroup K_1 (noting that c_2 does not depend on u_2 by assumption).

Thus the measurable map $\xi : u_2 \mapsto \Delta_{w_2} b_2(\,\cdot\,,u_2)\cdot c_2(\,\cdot\,)$ from K_2 to the Polish Abelian group $\mathcal{C}(K_1)$ of isomorphism classes of Borel maps up to almost-everywhere agreement actually almost surely takes values in the closed subgroup $\mathcal{E}(K_1)$. Let us also define another measurable map $\beta : K_2 \to \mathcal{C}(K_1)$ by $\beta(u_2) := b_2(\,\cdot\,,u_2)$.

If we now choose any $\theta \in K_2$ and take the difference under θ of the definition of ξ, then since c_2 is K_2-invariant we obtain

$$\Delta_{w_2}\Delta_\theta\beta = \Delta_\theta\xi.$$

This tells us that as members of $\mathcal{C}(K_1)$, $\Delta_\theta\beta(u_2)$ and $\Delta_\theta\beta(u_2w_2)$ almost surely differ only by a member of $\mathcal{E}(K_1)$. Since $\mathcal{E}(K_1) \leq \mathcal{C}(K_1)$ is a closed subgroup and so the quotient group carries a smooth Borel structure, and since R_{w_2} is ergodic on K_2, it follows that there are some fixed Borel map $f_\theta \in \mathcal{C}(K_2)$ and a Borel selection of eigenfunctions $u_2 \mapsto \zeta_\theta(u_2) \in \mathcal{E}(K_1)$ such that $\Delta_\theta\beta(u_2) = f_\theta \cdot \zeta_\theta(u_2)$, and moreover a simple measurable selection argument ensures that we can take these to vary Borel measurably in θ while still guaranteeing that this equation hold Haar-almost everywhere, so we may write instead $\Delta_\theta\beta(u_2) = f(\theta)\cdot\zeta(\theta, u_2)$.

It follows that if we define $\overline{\beta} : K_2 \to \mathcal{C}(K_1)/\mathcal{E}(K_1)$ to be the quotient of β and similarly for $\overline{f}$, then $\Delta_\theta\overline{\beta}(u_2) = \overline{f}(\theta)$. Therefore $\overline{f}$ is a homomorphism, since given θ and θ' we know that for almost every $u_2 \in K_2$ we have

$$\overline{f}(\theta) \cdot \overline{f}(\theta') = \Delta_\theta\overline{\beta}(u_2) \cdot \Delta_{\theta'}\overline{\beta}(u_2 \cdot \theta) = \Delta_{\theta\cdot\theta'}\overline{\beta}(u_2) = \overline{f}(\theta \cdot \theta'),$$

and hence $\overline{\beta}$ is an affine homomorphism (each up to modification on a negligible set).

We may therefore find some fixed function $h \in \mathcal{C}(K_1)$ such that if we write $\overline{h}$ for the image of h in $\mathcal{C}(K_1)/\mathcal{E}(K_1)$, define $\tilde{\beta}(u_2) := \beta(u_2) \cdot h$ and let $\overline{\tilde{\beta}}$ be its image under composition with the quotient map $\mathcal{C}(K_2) \to \mathcal{C}(K_2)/\mathcal{E}(K_1)$, then this $\overline{\tilde{\beta}}$ is a true homomorphism. Hence regarding it as a member of $\mathcal{Z}^1(K_1, \mathcal{C}(K_1)/\mathcal{E}(K_1))$ we have $d\overline{\tilde{\beta}} = 0$. However, this in turn tells us that the 2-cocycle $d\tilde{\beta}$ takes values in the closed subgroup $\mathcal{E}(K_1)$, endowed with the trivial action of K_2, which we note is continuously isomorphic to $\mathrm{S}^1 \times \widehat{K_1}$ under the multiplication map $(t, \chi) \mapsto t \cdot \chi$, so that $d\tilde{\beta}$ may be identified with a pair of 2-cocycles, one taking values in $\mathbb{T}$ and the other in $\widehat{K_1}$.

Step 2 We now bring Lemma A.6 to bear on this cocycle $d\tilde{\beta}$. Each K_i can be represented as an inverse limit of finite-dimensional groups, say as

$$(K_i, (q_{(m),i})_{m\geq 0}) = \lim_{m\leftarrow} \big((K_{(m),i})_{m\geq 0}, (q^{(m)}_{(k),i})_{m\geq k\geq 0}\big),$$

and correspondingly the group $\widehat{K_i}$ is the direct limit of the groups $\widehat{K_{(m),i}}$ under the embeddings given by composition with $q_{(m),i}$. From the continuity of $\mathrm{H}^2(\cdot,\cdot)$ given by Proposition A.3 it follows that $d\tilde{\beta}$ is cohomologous to a 2-cocycle that depends only on a finite-dimensional quotient group $K_{(m),2}$ of K_2, and takes values in the lift of some $\widehat{K_{(m),1}}$: that is, we can write

$$d\tilde{\beta} = d\rho_2 \cdot \kappa \circ q^{\times 2}_{(m),2}$$

for some $\rho_2 : K_2 \to \mathcal{E}(K_1)$ and 2-cocycle $\kappa : K_{(m),2} \times K_{(m),2} \to \mathcal{E}(K_{(m),1})$.

As the dual of a finite-dimensional Abelian group, $\widehat{K_{(m),1}}$ is finitely-generated and so the Structure Theorem for these identifies it with some direct product $\mathbb{Z}^D \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$. Hence we obtain similarly $\mathcal{E}(K_{(m),1}) \cong \mathbb{T} \times \mathbb{Z}^D \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ with trivial K_2-action, and so applying the relevant parts of Lemma A.6 to each coordinate we obtain that, by a further adjustment of ρ_2 if necessary, we can assume that κ takes the form

$$\kappa(u_2, v_2) = \kappa'(u_2, v_2) \cdot \prod_{j=1}^{k} \gamma_j^{\lfloor \{\chi_j(u_2)\} + \{\chi_j(v_2)\} \rfloor}$$

for some 2-cocycle $\kappa' : K_{(m),2} \times K_{(m),2} \to \mathrm{S}^1 \cdot (\widehat{K_{(m),1}})_{\mathrm{tor}}$ that depends only on a finite group quotient of $K_{(m),2}$ (where we write $(\widehat{K_{(m),1}})_{\mathrm{tor}}$ for the torsion subgroup of $\widehat{K_{(m),1}}$, which must in turn consist of those characters that are lifted from the maximal finite group quotient of $K_{(m),1}$), and finite lists $\gamma_1, \gamma_2, \ldots, \gamma_J \in \widehat{K_{(m),1}}$, $\chi_1, \chi_2, \ldots, \chi_J \in \widehat{K_{(m),2}}$.

Step 3 Consider the 2-cocycle

$$\prod_{j=1}^{J} \gamma_j^{\lfloor \{\chi_j(u_2)\} + \{\chi_j(v_2)\} \rfloor}$$

appearing in the above factorization. An explicit computation shows that this can be represented as the coboundary $d\beta'$ of the following $\mathcal{C}(K_1)$-valued 1-cochain:

$$\beta'(u_2)(u_1) = \prod_{j=1}^{J} \exp(2\pi\mathrm{i}\{\chi_j(u_2)\}\{\gamma_j(u_1)\}).$$

It follows that

$$\kappa' \circ q_{(m),2}^{\times 2} = \kappa \circ q_{(m),2}^{\times 2} \cdot \overline{d(\beta' \circ q_{(m),2})} = d(\tilde{\beta} \cdot \overline{\rho_2} \cdot \overline{\beta' \circ q_{(m),2}}),$$

so the lift of κ' to $K_2 \times K_2$ is a $\mathcal{C}(K_1)$-valued coboundary.

Step 4 Let us now write $r_{(m),i} : K_{(m),i} \twoheadrightarrow F_{(m),i}$ for the maximal finite group quotient of $K_{(m),i}$, whose kernel is just the identity connected component in $K_{(m),i}$. We have seen that κ' factorizes through $r_{(m),2} \times r_{(m),2}$ and takes values in $\mathcal{E}(r_{(m),1})$.

Also, from the above we have that $\kappa' \circ q_{(m),2}^{\times 2}$ is a $\mathcal{C}(K_1)$-valued coboundary. Since on the one hand $r_{(m),1} \circ q_{(m),1} : K_1 \twoheadrightarrow F_{(m),1}$ has finite image, and so its fibres

all have individually positive measure, and on the other hand our action of K_2 on $\mathcal{C}(K_1)$ is trivial, simply by choosing a representative point from each fibre of $r_{(m),1}\circ q_{(m),1}$ at random and sampling $\tilde{\beta}\cdot\overline{\rho_2}\cdot\overline{\beta'\circ q_{(m),2}}$ at those points we deduce that $\kappa'\circ q_{(m),2}^{\times 2}$ is actually the coboundary of some $\mathcal{C}(r_{(m),1})$-valued 1-cochain.

We will now argue further that, possibly after a finite further increase in m, it must be the $\mathcal{C}(r_{(m),1})$-valued coboundary of some 1-cochain that depends only on coordinates in $F_{(m),2}$. Indeed, this also follows directly from Lemma A.6, since in view of the triviality of the action we can simply write $\mathcal{C}(r_{(m),1})\cong\mathbb{T}^{\oplus F_{(m),1}}$ as K_2-modules, and for each of these finitely many copies of $\mathbb{T}$ Part 3 of Lemma A.6 gives some $m'\geq m$ such that $\kappa'(\,\cdot\,,\,\cdot\,)(x)$ regarded as a $\mathbb{T}$-valued cocycle is a coboundary upon lifting only up as far as $F_{(m'),2}\times F_{(m'),2}$. Taking the maximum of the m' so obtained for different $x\in F_{(m),1}$ gives the result.

Hence after passing to a suitably-enlarged value of m if necessary we can express $\kappa' = d(\alpha\circ r_{(m),2})$ for some $\alpha : F_{(m),2}\to\mathcal{C}(r_{(m),1})$, which we may of course alternatively interpret as a S^1-valued function that factorizes through $r_{(m),1}\times r_{(m),2}$.

Step 5 We have now represented the whole of κ as the $\mathcal{C}(K_{(m),1})$-valued coboundary: $d((\alpha\circ(r_{(m),2}\circ q_{(m),2}))\cdot(\beta'\circ q_{(m),2})))$ where

$$\beta'(u_2)(u_1) = \prod_{j=1}^{k}\exp(2\pi\mathrm{i}\{\chi_j(u_2)\}\{\gamma_j(u_1)\})$$

and α takes values in $\mathcal{C}(r_{(m),1})$.

Let us now write α and β' for the lifts of these cochains to K_2 to lighten notation, omitting the compositions with $q_{(m),2}$. Putting this factorization together with the definition of κ we have $d\tilde{\beta} = d(\rho_2\cdot\alpha\cdot\beta')$, and hence $d(\tilde{\beta}\cdot\overline{\rho_2\cdot\alpha\cdot\beta'}) = 0$ so that $\tilde{\beta}\cdot\overline{\rho_2\cdot\alpha\cdot\beta'} : K_2\to\mathcal{C}(K_1)$ is a Borel homomorphism. From this a simple inspection of the behaviour of the map $u_2\mapsto(\tilde{\beta}\cdot\overline{\rho_2\cdot\alpha\cdot\beta'})(u_2)(u_1)$ pointwise for almost every u_1 (formally, we are using Moore's treatment of direct-integral cohomology groups in Theorem 2 of [24]) indicates that there is some $\rho_1' : K_1\times K_2$ such that $\rho_1'(u_1,\,\cdot\,)$ is almost always a member of $\mathcal{E}(K_2)$ and

$$(\tilde{\beta}\cdot\overline{\rho_2\cdot\alpha\cdot\beta'})(u_2)(u_1) = \rho_1'(u_1,u_2)$$

almost everywhere.

Re-arranging this and recalling that $\tilde{\beta}(u_1,u_2) = b_2(u_1,u_2)h(u_1)$, we have obtained a factorization

$$\begin{aligned} b_2(u_1,u_2) &= \overline{h(u_1)}\cdot\alpha(u_1,u_2)\cdot\rho_1'(u_1,u_2)\cdot\rho_2(u_1,u_2)\cdot b_2'(u_1,u_2) \\ &= \alpha(u_1,u_2)\cdot\rho_1(u_1,u_2)\cdot\rho_2(u_1,u_2)\cdot b_2'(u_1,u_2) \end{aligned}$$

where $\alpha : K_1 \times K_2 \to \mathrm{S}^1$ factorizes through the finite quotient $r_{(m),1} \times r_{(m),2}$, $\rho_1(u_1,u_2) := \overline{h(u_1)} \cdot \rho_1'(u_1,u_2)$ has the property that that $\underline{\rho_1(u_1},\,\cdot\,)$ is a member of $\mathcal{E}(K_2)$ for Haar-almost every $u_1 \in K_1$ (with each value $\overline{h(u_1)}$ interpreted simply as a constant function of u_2), ρ_2 has the symmetric property and b_2' is of the form

$$b_2'(u_1,u_2) = \exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J}\{\gamma_j(u_1)\}\{\chi_j(u_2)\}\Big)$$

This gives us the asserted factorization of b_2.

Step 6 Our last step is to turn the above factorization into a suitable cohomology for each of c_1 and c_2.

To do this we now difference the factorization of b_2 obtained above with respect to w_1 and w_2 and insert the result back into our original commutativity equation for σ. This becomes

$$\begin{aligned}&\Delta_{w_2}c_1(u_2)\\ &= (\Delta_{w_1}\Delta_{w_2}\alpha(u_1,u_2))(\Delta_{w_1}\rho_1(u_1,w_2))(\Delta_{w_2}\rho_2(w_1,u_2))(\Delta_{w_1}\Delta_{w_2}b_2(u_1,u_2))\cdot\Delta_{w_1}c_2(u_1).\end{aligned}$$

On the other hand, we can compute explicitly that

$$\begin{aligned}\Delta_{w_1}\Delta_{w_2}b_2'(u_1,u_2) &= \Delta_{w_1}\Big(\prod_{j=1}^{J}\exp(2\pi\mathrm{i}\{\gamma_j(u_1)\}(\{\chi_j(u_2+w_2)\} - \{\chi_j(u_2)\})\Big)\\
&= \Delta_{w_1}\Big(\prod_{j=1}^{J}\exp(2\pi\mathrm{i}\{\gamma_j(u_1)\}(\{\chi_j(w_2)\} - \lfloor\{\chi_j(u_2)\} + \{\chi_j(w_2)\}\rfloor))\Big)\\
&= \Delta_{w_1}\Big(\prod_{j=1}^{J}\exp(2\pi\mathrm{i}\{\gamma_j(u_1)\}\{\chi_j(w_2)\})\cdot\prod_{j=1}^{J}\gamma_j(u_1)^{-\lfloor\{\chi_j(u_2)\}+\{\chi_j(w_2)\}\rfloor}\Big)\\
&= \prod_{j=1}^{J}\mathrm{e}^{2\pi\mathrm{i}\{\gamma_j(w_1)\}\{\chi_j(w_2)\}}\cdot\prod_{j=1}^{J}\gamma_j(w_1)^{-\lfloor\{\chi_j(u_2)\}+\{\chi_j(w_2)\}\rfloor}\cdot\prod_{j=1}^{J}\chi_j(w_2)^{-\lfloor\{\gamma_j(u_1)\}+\{\gamma_j(w_1)\}\rfloor}.\end{aligned}$$

Also, we have

$$\begin{aligned}&\prod_{j=1}^{J}\gamma_j(w_1)^{-\lfloor\{\chi_j(u_2)\}+\{\chi_j(w_2)\}\rfloor}\\
&= \exp\Big(-2\pi\mathrm{i}\sum_{j=1}^{J}\{\gamma_j(w_1)\}\{\chi_j(w_2)\}\Big)\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J}\gamma_j(w_1)(\{\chi_j(u_2+w_2)\}-\{\chi_j(u_2)\})\Big)\end{aligned}$$

and similarly for $\prod_{j=1}^J \chi_j(w_2)^{-\lfloor\{\gamma_j(u_1)\}+\{\gamma_j(w_1)\}\rfloor}$, so we can write the above factorization as

$$\Delta_{w_1}\Delta_{w_2}b_2'(u_1,u_2) = \big(\mathrm{constant}\big)\cdot\Delta_{w_2}f_1(u_2)\cdot\Delta_{w_1}f_2(u_1)$$

with

$$f_1(u_2) := \exp\Big(2\pi\mathrm{i}\sum_{j=1}^J\{\gamma_j(w_1)\}\{\chi_j(u_2)\}\Big)$$

$$\text{and}\qquad f_2(u_1) := \exp\Big(2\pi\mathrm{i}\sum_{j=1}^J\{\chi_j(w_2)\}\{\gamma_j(u_1)\}\Big).$$

It follows that we may re-arrange the commutativity condition to deduce that both

$$\Delta_{w_2}(c_1(u_2)\cdot\overline{\rho_2(w_1,u_2)}\cdot\overline{f_1(u_2)})$$

and

$$\Delta_{w_1}(c_2(u_1)\cdot\rho_1(u_1,w_2)\cdot f_2(u_1))$$

must actually factorize through the finite quotient of $K_1\times K_2$ under $r_{(m),1}\times r_{(m),2}$.

Since for any $n\geq 1$ we can form

$$\Delta_{w_2^n}(c_1(u_2)\cdot\overline{\rho_2(w_1,u_2)}\cdot\overline{f_1(u_2)})$$

by multiplying translates of

$$\Delta_{w_2}(c_1(u_2)\cdot\overline{\rho_2(w_1,u_2)}\cdot\overline{f_1(u_2)}),$$

and R_{w_2} acts ergodically on K_2, it follows that we can find some $n\geq 1$ such that $r_{(m),2}(w_2^n) = 1$, and thus that the above condition tells us that in each ergodic component of $R_{w_2^n}$ acting on K_2 the function

$$\Delta_{w_2^n}(c_1(u_2)\cdot\overline{\rho_2(w_1,u_2)}\cdot\overline{f_1(u_2)})$$

is constant, and hence that

$$c_1(u_2)\cdot\overline{\rho_2(w_1,u_2)}\cdot\overline{f_1(u_2)}$$

must an eigenfunction within each of these ergodic components. Calling this function $g_1(u_2)$, and obtaining similarly $g_2(u_1)$, one last re-arrangement gives that

$$c_1(u_2) = \rho_2(w_1,u_2)\cdot f_1(u_2)\cdot g_1(u_2) = \Delta_{w_1}\rho_2(u_1,u_2)\cdot f_1(u_2)\cdot g_1(u_2)$$

and

$$c_2(u_1) = \overline{\rho_1(w_1,u_2)} \cdot \overline{f_2(u_1)} \cdot g_2(u_1) = \Delta_{w_2}\overline{\rho_1(u_1,u_2)} \cdot \overline{f_2(u_1)} \cdot g_2(u_1).$$

Since the function $g_1(u_2)$ is an eigenfunction within each coset of some finite-index subgroup of K_2, it follows that we may write g_1 in the form

$$g_1(u_2) = \alpha_i(u_2) \prod_{j'=1}^{J'} \chi'_j(u_2)^{a_j(u_2)} = \alpha_i(u_2) \exp\Big(2\pi\mathrm{i} \sum_{j'=1}^{J'} a_j(u_2)\{\chi'_j(u_2)\}\Big)$$

for some maps $\alpha_i : K_2 \to \mathrm{S}^1$ and $a_j : K_2 \to \mathbb{Z}$ that factorize through some finite quotient group of K_2, and some additional characters $\chi'_j \in \widehat{K_2}$. Combining this with the explicit form obtained above for $f_1(u_2)$ and noting that $\Delta_{w_1}\rho_2(u_1,u_2)$ is an R_{w_1}-coboundary, we see that we have put $c_1(u_2)$ explicitly into the desired form, and similarly for $c_2(u_1)$. This completes the proof of Proposition 2.1. $\square$

3 Proof of the main theorem

We now turn to Theorem 1.1:

Theorem. *If $T_1, T_2 : \mathbb{Z} \curvearrowright (X,\mu)$ commute then the averages*

$$S_N(f_1,f_2) := \frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T_1^{n^2})(f_2 \circ T_1^{n^2}T_2^n)$$

converge in $L^2(\mu)$ as $N \to \infty$ for any $f_1, f_2 \in L^\infty(\mu)$.

The proof proceeds through a sequence of three reductions to progressively simpler classes of polynomial average, each obtained by deriving different consequences from some invocation of the van der Corput estimate. After the third reduction we will reach a family of averages to which known results can be applied more-or-less directly.

In rough outline, our first reduction amounts to an identification of characteristic factors for these polynomial averages in some pleasant extension, so that we may assume the functions f_1 and f_2 take a special form in terms of these factors. This use of characteristic factors is another outing for what is now the standard approach to such questions. It is for this first step that we will need the result for linear averages of Theorem 1.2. In fact, we will need just a little more versatility

than is contained in Theorem 1.2 as stated, but which follows at once from combining that theorem with the following immediate consequence of the definition of a characteristic tuple of factors (see Lemma 4.3 in [4]):

Lemma 3.1. *For any factor $\xi : \mathbf{X} \to \mathbf{Y}$ the triple $(\xi, \mathrm{id}_X, \mathrm{id}_X)$ is characteristic for the nonconventional averages*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T^{n\mathbf{p}_1})(f_2 \circ T^{n\mathbf{p}_2})(f_3 \circ T^{n\mathbf{p}_3}), \qquad f_1, f_2, f_3 \in L^\infty(\mu),$$

if and only if the triple $(\mathrm{id}_X, \xi, \mathrm{id}_X)$ is characteristic for the nonconventional averages

$$\frac{1}{N}\sum_{n=1}^{N}(f_0 \circ T^{-n\mathbf{p}_j})(f_1 \circ T^{n(\mathbf{p}_1-\mathbf{p}_j)})(f_k \circ T^{n(\mathbf{p}_k-\mathbf{p}_j)}), \qquad f_0, f_1, f_k \in L^\infty(\mu),$$

whenever $\{j,k\} = \{2,3\}$. □

Corollary 3.2. *In the statement of Theorem 1.2 we may instead let the target system of η be a $(\mathbf{p}_1 - \mathbf{p}_2, m_{13}(\mathbf{p}_1 - \mathbf{p}_3), m_1\mathbf{p}_1)$-directional CL-system or a $(\mathbf{p}_1 - \mathbf{p}_3, m_{12}(\mathbf{p}_1 - \mathbf{p}_2), m_1\mathbf{p}_1)$-directional CL-system.* □

Our use for Theorem 1.2 and Lemma 3.2 will be to prove an even more precise description of a characteristic pair of factors for our nonconventional quadratic averages, by considering a whole family of triple linear averages that arise from those quadratic averages through an appeal to the van der Corput estimate, and then examining the possible joint distribution of the characteristic factors for those different triple linear averages inside the overall system. The result of this step will be Theorem 1.3.

The second reduction then follows quite quickly and uses similar ideas: after simplifying the averages S_N for functions measurable with respect to the new characteristic pair of factors and re-arranging slightly, a new sequence of averages emerges to which another appeal to Theorem 1.2 and the resulting description of the Furstenberg self-joining gives a further simplification.

The proof is completed through a closer examination of some functions measurable with respect to a $(h\mathbb{Z}^2, (h,0), (0,h))$-directional CL-system for some h. This is heavily based on an older approach of Host and Kra [16] to the study of the triple linear nonconventional averages associated to three powers of a single transformation that does not need the exact picture in terms of nilsystems, which was not available at the time of that paper. It amounts to a way of using directly the

combined cocycle equation arising from the Mackey data inside the Furstenberg self-joining of our system. This leads to a classification of the polynomial averages output by the second reduction into two cases. In the first case we can show they tend to 0 in $L^2(\mu)$, and in the second we will find that they can eventually be re-written simply as a more classical sequence of weighted ergodic averages, for which mean convergence is known.

3.1 Directional CL-systems

We are now ready to introduce the 'directional CL-systems' that are the main new ingredient that appear in Theorem 1.2. In this subsection we will define these systems and establish some of their basic properties.

Directional CL-cocycles are characterized by the existence of solutions to some natural 'directional' analogs of the classic Conze-Lesigne equations among cocycles ([9, 21]). Let us first introduce these equations, and then the class of cocycles that they specify.

Definition 3.3 (Directional Conze-Lesigne equations)**.** *Suppose that A and Z are compact metrizable Abelian groups, $K \leq Z$ a closed subgroup and $\tau : Z \to A$ a Borel map. Then another Borel map $b : Z \to A$* ***satisfies the directional Conze-Lesigne equation E**(u, v, K, τ) for some $u, v \in Z$ if there is a Borel map $c : Z/K \to A$ such that*

$$\Delta_u \tau(z) = \Delta_v b(z) \cdot c(z \cdot K) \qquad \textit{for } m_Z\textit{-a.e. } z.$$

It is clear that this c is then uniquely determined. We refer to b as a ***solution*** *of the equation $E(u, v, K, \tau)$ and to c as the* ***one-dimensional auxiliary*** *of b in this equation. This is the classical Conze-Lesigne equation in case $K = G$.*

Although we have formulated the above definition for cocycles into an arbitrary compact Abelian target group A, for technical reasons we will use this equation only for cocycles into S^1.

Remark on notation Similarly to [5], we will henceforth write $(Z_\star, m_{Z_\star}, \phi_\star)$ to denote a $\mathbb{Z}^2$-system whose underlying space is the direct integral of some measurably-varying family of compact Abelian groups $Z_\star$, indexed by some other standard Borel probability space (S, ν) on which the action is trivial, with the overall action a fibrewise rotation defined by a measurable selection for each fibre Z_s of a dense homomorphism $\phi_s : \mathbb{Z}^2 \to Z_s$: writing R_ϕ for this action, it is given by

$$R_\phi^{\mathbf{n}}(s, z) := (s, z \cdot \phi_s(\mathbf{n})) \qquad \text{for } s \in S,\ z \in Z_s \text{ and } \mathbf{n} \in \mathbb{Z}^2.$$

We will refer to such a system as a **direct integral of ergodic group rotations** and to (S,ν) as its **invariant base space**. Sometimes we omit the base space (S,ν) from mention completely, since once again the forthcoming arguments will all effectively be made fibrewise, just taking care that all newly-constructed objects can still be selected measurably. In particular, we will often write just $Z_\star$ in place of $S \ltimes Z_\star$. $\lhd$

Definition 3.4 (Directional CL-cocycles)**.** *Suppose that* $\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3 \in \mathbb{Z}^2$*, that* $(Z_\star, m_{Z_\star}, \phi_\star)$ *is a direct integral of ergodic* $\mathbb{Z}^2$*-group rotations with invariant base space* (S,ν)*, and that* $A_\star$ *is motionless compact metrizable Abelian group data over* $(Z_\star, m_{Z_\star}, \phi_\star)$*.*

A cocycle-section $\tau : \mathbb{Z}^2 \times Z_\star \to A_\star$ *over the fibrewise rotation action* R_ϕ *is an* $(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)$***-directional CL-cocycle over*** R_ϕ *if for every* R_ϕ*-invariant measurable selection of characters* $\chi_\star \in \widehat{A_\star}$ *we have that*

- *for every* R_ϕ*-invariant measurable selection* $u_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}$ *there is a Borel map* $b : S \ltimes Z_\star \to \mathrm{S}^1$*, denoted by* $b_\star$*, such that* b_s *solves the equation* $E(u_s, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n}_1,\,\cdot\,)|_{Z_s})$ *for* ν*-almost every* s*, and*
- *for every* R_ϕ*-invariant measurable selection* $v_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}$ *there is a Borel map* $b_\star : S \ltimes Z_\star \to \mathrm{S}^1$ *that solves the equation* $E(v_s, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_2)}, \chi_s \circ \tau(\mathbf{n}_1,\,\cdot\,)|_{Z_s})$ *for* ν*-almost every* s*.*

Given a subgroup $\Gamma \leq \mathbb{Z}^2$*,* τ *is a* $(\Gamma,\mathbf{n}_2,\mathbf{n}_3)$***-directional CL-cocycle over*** R_ϕ *if for every* R_ϕ*-invariant measurable selection of characters* $\chi_\star \in \widehat{A_\star}$ *we have that*

- *for every* R_ϕ*-invariant measurable selection* $u_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}$ *there is a Borel map* $b_\star : S \ltimes Z_\star \to \mathrm{S}^1$ *that* simultaneously *solves the equations* $E(u_s, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n}_1,\,\cdot\,)|_{Z_s})$*,* $\mathbf{n}_1 \in \Gamma$*, for* ν*-almost every* s*, and*
- *for every* R_ϕ*-invariant measurable selection* $v_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}$ *there is a Borel map* $b_\star : S \ltimes Z_\star \to \mathrm{S}^1$ *that* simultaneously *solves the equations* $E(v_s, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_2)}, \chi_s \circ \tau(\mathbf{n}_1,\,\cdot\,)|_{Z_s})$*,* $\mathbf{n}_1 \in \Gamma$*, for* ν*-almost every* s*.*

In the above situation we will usually write more briefly that

> 'for every $\chi_\star \in \widehat{A_\star}$ and $u_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}$, the map $b_\star : Z_\star \to \mathrm{S}^1$ is a solution to the equations $\mathrm{E}(u_\star, \phi_\star(\mathbf{n}_1), \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}, \chi_\star \circ \tau(\mathbf{n}_1,\,\cdot\,))$',

and similarly for the other equations (note, in particular, that the restriction of $\tau(\mathbf{n}_1, \cdot\,)$ to the relevant fibre $Z_\star$ is left to the understanding).

Lemma 3.5. *If $\Gamma \leq \mathbb{Z}^2$ is a subgroup generated by a subset $F \subset \mathbb{Z}^2$ then a cocycle-section $\tau : \mathbb{Z}^2 \times Z_\star \to A_\star$ is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ for every $\mathbf{n}_1 \in F$ if the simultaneous solutions required above exist only for all of the families of equations*

$$\bigvee_{\mathbf{n}_1 \in F} \mathrm{E}(u_\star, \phi_\star(\mathbf{n}_1), \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}, \chi_\star \circ \tau(\mathbf{n}_1, \cdot\,))$$

and

$$\bigvee_{\mathbf{n}_1 \in F} \mathrm{E}(v_\star, \phi_\star(\mathbf{n}_1), \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}, \chi_\star \circ \tau(\mathbf{n}_1, \cdot\,)).$$

Proof This follows from the simple property of the directional Conze-Lesigne equations that if, say, $u \in \overline{\phi_s(\mathbb{Z}\mathbf{n}_2)}$, $\mathbf{n}, \mathbf{n}' \in F$ and b solves the equations

$$\mathrm{E}(u, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n}_1, \cdot\,)|_{Z_s})$$

for both $\mathbf{n}_1 = \mathbf{n}$ and $\mathbf{n}'$ with respective one-dimensional auxiliaries c and c', then

$$\begin{aligned}
\Delta_u \tau(\mathbf{n} + \mathbf{n}', z) &= \Delta_u \tau(\mathbf{n}, z + \phi_s(\mathbf{n}')) \cdot \Delta_u \tau(\mathbf{n}', z) \\
&= \Delta_{\mathbf{n}} b(z + \phi_s(\mathbf{n}')) \cdot \Delta_{\mathbf{n}'} b(z) \\
&\qquad \cdot c((z + \phi_s(\mathbf{n}')) \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}) \cdot c'(z \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}) \\
&= \Delta_{\mathbf{n}+\mathbf{n}'} b(z) \cdot c''(z \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)})
\end{aligned}$$

at m_{Z_s}-a.e. z, where c'' is the obvious product function formed from c and c'. Therefore b is also a solution to

$$\mathrm{E}(u, \phi_s(\mathbf{n} + \mathbf{n}'), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n} + \mathbf{n}', \cdot\,)|_{Z_s}).$$

A similar argument shows that it also solves

$$\mathrm{E}(u, \phi_s(-\mathbf{n}), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(-\mathbf{n}, \cdot\,)|_{Z_s}),$$

and so in fact it applies to the whole subgroup Γ, as required. $\square$

Remark For the above proof it would clearly not be enough to demand that the equations $\mathrm{E}(u_s, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n}_1, \cdot\,)|_{Z_s})$ for different $\mathbf{n}_1 \in \Gamma$ have solutions separately. The requirement of simultaneous solutions when working with $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycles will be very important later precisely so that we can use similar manipulations again. $\lhd$

With the above preparations behind us, we can now define our new class of systems itself.

Definition 3.6 (Directional CL-extensions and systems)**.** *If $\mathbf{X}$ is a $\mathbb{Z}^2$-system, $(Z_\star, m_{Z_\star}, \phi_\star)$ is a direct integral of ergodic $\mathbb{Z}^2$-group rotations and $\pi : \mathbf{X} \to (Z_\star, m_{Z_\star}, \phi_\star)$ is a factor map, then $\mathbf{X}$ is an $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$-**directional CL-extension** of $(Z_\star, m_{Z_\star}, \phi_\star)$ through π if it can be coordinatized as $(Z_\star, m_{Z_\star}, \phi_\star) \ltimes (A_\star, m_{A_\star}, \tau)$ with π the canonical factor and τ an $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ. More loosely, $\mathbf{X}$ is an $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$-**directional CL-system** if it is an $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-extension of some factor that is a direct integral of group rotations, and then any suitable choice for this group-rotation factor is a **base** for $\mathbf{X}$.*

*If $\Gamma \leq \mathbb{Z}^2$ then $\mathbf{X}$ is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-**directional CL-extension** of $(Z_\star, m_{Z_\star}, \phi_\star)$ if the above coordinatization is possible with τ a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle.*

We will write $\mathsf{Z}^{\Gamma,\mathbf{n}_2,\mathbf{n}_3}_{\mathrm{dCL}}$ for the class of $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-systems, and generally write this as $\mathsf{Z}^{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3}_{\mathrm{dCL}}$ if $\Gamma = \mathbb{Z}\mathbf{n}_1$.

The elementary properties of directional CL-cocycles follow easily from the directional Conze-Lesigne equations.

Lemma 3.7. *Suppose that $\pi : (\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \to (Z_\star, m_{Z_\star}, \phi_\star)$ is a tower of direct integrals of $\mathbb{Z}^2$-group rotations. Then*

(1) if $\tau_1 : \mathbb{Z}^2 \times Z_\star \to A_\star$ is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ then $\tau_1 \circ \pi$ is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over $R_{\tilde{\phi}}$;

(2) if $\tau_2 : \mathbb{Z}^2 \times Z_\star \to A_\star$ is another $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ then $\tau_1 \cdot \tau_2$ is also a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ;

(3) $(A_{(m),\star})_{m\geq 1}$, $(\Phi^{(m)}_{(k),\star})_{m\geq k\geq 0}$ is a motionless measurable family of inverse sequences of compact Abelian groups over $(Z_\star, m_{Z_\star}, \phi_\star)$ with inverse limit family $A_{(\infty),\star}$, $(\Phi_{(m),\star})_{m\geq 0}$ (which is clearly still measurable), and $\tau_{(m)} : \mathbb{Z}^2 \times Z_\star \to A_{(m),\star}$ is a family of $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycles over R_ϕ satisfying the consistency equations $\tau_{(k)} = \Phi^{(m)}_{(k),\star} \circ \tau_{(m)}$ for $m \geq k \geq 0$, then the resulting inverse limit cocycle $\tau_{(\infty)} : \mathbb{Z}^2 \times Z_\star \to A_{(\infty),\star}$ is also a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle.

Proof The first two parts follow immediately from lifting and multiplying solutions to the directional Conze-Lesigne equations, since π must map each group rotation fibre of $(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$ onto a group rotation fibre of $(Z_\star, m_{Z_\star}, \phi_\star)$ via a measurably-varying continuous affine epimorphism.

For the third part, first recall that by construction any character on an inverse limit of compact Abelian groups factorizes through some finite level of the inverse sequence. This implies that for any measurable selection of characters $\chi_\star \in \widehat{A_{(\infty),\star}}$ we can find a measurable selection of positive integers $m_\star$ such that $\chi_\star$ factorizes through $\Phi_{(m_\star),\star} : A_{(\infty),\star} \to A_{(m_\star),\star}$ almost surely (so $\chi_\star \circ \tau_{(\infty)} = \chi'_\star \circ \tau_{(m_\star)}$ for some measurable selection of characters satisfying $\chi_\star = \chi'_\star \circ \Phi_{(m_\star),\star}$). Now we may simply call on the solutions to the directional Conze-Lesigne equations for this $\tau_{(m_\star)}$ within each level set of the map $m_\star$, to see that these patch together to give solutions to the directional Conze-Lesigne equations for $\tau_{(\infty)}$. Note that this last step illustrates the usefulness of defining directional CL-cocycles in terms of the behaviour of their compositions with characters, rather than directly, as discussed above. $\square$

Now suppose that $(Z_{i,\star}, m_{Z_{i,\star}}, \phi_{i,\star})$ are direct integrals of ergodic $\mathbb{Z}^2$-group rotations for $i = 1, 2$ and that θ is a joining of them. Then we may form the measurably-varying family of compact Abelian groups $Z_{1,\star} \times Z_{2,\star}$ simply by taking the product of the underlying invariant base spaces (S_i, ν_i), and then taking the products of the two fibres of each pair of index points (s_1, s_2) from those spaces; and similarly we can define the obvious homomorphism $(\phi_{1,s_1}, \phi_{2,s_2}) : \mathbb{Z}^2 \to Z_{1,s_1} \times Z_{2,s_2}$ above each such pair of index points. Now a simple application of the non-ergodic Mackey Theorem (Theorem 2.1 in [5]) shows that θ decomposes further into a direct integral of Haar measures on the cosets of the measurably-varying family of subgroups

$$\overline{\{(\phi_{1,s_1}(\mathbf{n}), \phi_{2,s_2}(\mathbf{n})) :\ \mathbf{n} \in \mathbb{Z}^2\}} \leq Z_{1,s_1} \times Z_{2,s_2},$$

and so the joined system $(Z_{1,\star} \times Z_{2,\star}, \theta, (\phi_{1,\star}, \phi_{2,\star}))$ can also be expressed as a direct integral of ergodic $\mathbb{Z}^2$-group rotations (although the ergodic fibres may be strictly smaller than $Z_{1,\star} \times Z_{2,\star}$, and the underlying invariant index space correspondingly larger).

Combined with the above lemma this implies that given two $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-extensions $\pi_i : \mathbf{X}_i \to (Z_{i,\star}, m_{Z_{i,\star}}, \phi_{i,\star})$ and any joining θ as above, the lift of θ to a relatively independent joining λ of $\mathbf{X}_1$ and $\mathbf{X}_2$ gives a joint system that is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-extension of $(Z_{1,\star} \times Z_{2,\star}, \theta, (\phi_{1,\star}, \phi_{2,\star}))$. This will be an important observation for us when combined with the following proposition.

Proposition 3.8. *Suppose that $\pi : \mathbf{X} = (X, \mu, T) \to (Z_\star, m_{Z_\star}, \phi_\star)$ is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-extension, and that $(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$ is another direct integral of ergodic $\mathbb{Z}^2$-group rotations which can be located into a tower of systems*

$$\mathbf{X} \xrightarrow{\tilde{\pi}} (\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \xrightarrow{\alpha} (Z_\star, m_{Z_\star}, \phi_\star)$$

so that $\tilde{\pi}$ is a relatively ergodic extension. Then $\mathbf{X}$ is also a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-extension of $(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$.

Proof This breaks into two steps.

Step 1 We first show that the result holds when $\tilde{\pi} = \pi \vee \zeta_0^T$ (so $(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$ is simply a coordinatization of the factor of $\mathbf{X}$ generated by the base copy of $(Z_\star, m_{Z_\star}, \phi_\star)$ and the overall invariant factor — this is easily seen to be another direct integral of ergodic group rotations, with the same fibres as $(Z_\star, m_{Z_\star}, \phi_\star)$ but possibly an enlargement of the invariant base system). This is the smallest possible choice that gives $\tilde{\pi}$ relatively ergodic. Let (S, ν) be the invariant base space underlying $(Z_\star, m_{Z_\star}, \phi_\star)$.

Suppose that $\tau : \mathbb{Z}^2 \times Z_\star \to A_\star$ is the $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ corresponding to some coordinatization of π. In this case the non-ergodic Mackey Theorem gives a precise coordinatization of $\tilde{\pi}$: there are a motionless family of closed subgroups $K_\star \leq A_\star$ and a measurable section $\rho : S \ltimes Z_\star \to A_\star$ such that $\tilde{\pi}$ can be coordinatized by the factor map

$$(S \ltimes Z_\star) \ltimes A_\star \to S \ltimes (A_\star/K_\star) : ((s,z),a) \mapsto (s, a \cdot \rho(s,z) \cdot K_{(s,z)}),$$

and so $\pi \vee \zeta_0^T$ in turn is coordinatized by

$$(S \ltimes Z_\star) \ltimes A_\star \to (S \ltimes Z_\star) \ltimes (A_\star/K_\star) : ((s,z),a) \mapsto ((s,z), a \cdot \rho(s,z) \cdot K_{(s,z)}).$$

If we now simply re-coordinatize π by fibrewise rotations by ρ, then τ is replaced by $\tau' := \tau \cdot \Delta_\phi \rho$ so this now almost surely takes values in $K_\star$, and this leads to an explicit recoordinatization of the extension $\pi \vee \zeta_0^T$ as

$$\begin{tikzcd}
\mathbf{X} \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "\pi\vee\zeta_0^T"'] & (Z_\star \ltimes (A_\star/K_\star), m_{Z_\star \ltimes (A_\star/K_\star)}, (\phi_\star, 1_{A_\star/K_\star})) \ltimes (K_\star, m_{K_\star}, \tau') \arrow[d, "\text{canonical}"] \\
& (Z_\star \ltimes (A_\star/K_\star), m_{Z \ltimes (A_\star/K_\star)}, (\phi_\star, 1_{A_\star/K_\star}))
\end{tikzcd}$$

(where we again abbreviate $S \ltimes Z_\star$ to $Z_\star$). In this diagram the base system $(Z_\star \ltimes (A_\star/K_\star), m_{Z_\star \ltimes (A_\star/K_\star)}, (\phi_\star, 1_{A_\star/K_\star}))$ is expressed as a direct integral of not-necessarily ergodic group rotations — indeed, the homomorphisms $\mathbf{n} \mapsto (\phi_s(\mathbf{n}), 1_{A_s/K_s})$ cannot have dense image unless $K_s = A_s$ — but by cutting down the fibres and enlarging the invariant base system as previously it may clearly be re-coordinatized as a direct integral of ergodic group rotations with the same fibres $Z_\star$ as originally.

Since τ' depends only on the factor $Z_\star \ltimes (A_\star/K_\star) \to Z_\star$ (since this is true of τ and ρ), it suffices to show that τ', like τ, admits solutions to all the relevant directional Conze-Lesigne equations. If $\chi_\star \in \widehat{K_\star}$ is a measurable selection of characters then we can extend each χ_s to a character on the whole of A_s which we also denote by χ_s (it is classical that this is always possible; see, for instance, Theorem 24.12 of Hewitt and Ross [14]), and a simple appeal to the Measurable Selector Theorem promises that we can choose these extensions so as still to form a measurable family. Now if $\mathbf{n} \in \Gamma$, $u \in \overline{\phi_s(\mathbb{Z}\mathbf{n}_2)}$ for some s and b is a solution to the equation $\mathrm{E}(u, \phi_s(\mathbf{n}_1), \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}, \chi_s \circ \tau(\mathbf{n},\,\cdot\,)|_{Z_s})$ with one-dimensional auxiliary c, then we check at once that $b' := b \cdot \Delta_u(\chi_s \circ \rho|_{Z_s})$ satisfies

$$\begin{aligned}
&\Delta_{\phi_s(\mathbf{n})} b'(z) \cdot c(z \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}) \\
&= \Delta_{\phi_s(\mathbf{n})}\Delta_u(\chi_s \circ \rho|_{Z_s}) \cdot \Delta_{\phi_s(\mathbf{n})} b(z) \cdot c(z \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}) \\
&= \Delta_{\phi_s(\mathbf{n})}\Delta_u(\chi_s \circ \rho|_{Z_s}) \cdot \Delta_u(\chi_s \circ \tau(\mathbf{n},\,\cdot\,)|_{Z_s}) \\
&= \Delta_u(\chi_s \circ \tau'(\mathbf{n},\,\cdot\,)|_{Z_s}).
\end{aligned}$$

Performing this procedure fibrewise on the Borel map $b_\star$ that gives a solution for a measurable selection $u_\star$ clearly gives a new Borel map $b'_\star$ as the new solution, as required.

Step 2 We now prove the general case. In fact this makes very little appeal to the exact structure of the system $(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$.

By Step 1 we can replace $\pi : \mathbf{X} \to (Z_\star, m_{Z_\star}, \phi_\star)$ by a suitable coordinatization of $\pi \vee \zeta_0^T$ if necessary, and so suppose that π itself is relatively ergodic. Suppose again that $\tau : \mathbb{Z}^2 \times Z_\star \to A_\star$ is the $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-cocycle over R_ϕ of a coordinatization of π. Clearly $\alpha : (\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \to (Z_\star, m_{Z_\star}, \phi_\star)$ is also a relatively ergodic Abelian isometric extension, so these two direct integrals of ergodic group rotations have the same underlying invariant base space, and since now both π and α are relatively ergodic the Relative Factor Structure Theorem (Theorem 2.5 in [5]) applied to the triangle

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\tilde{\pi}"] \arrow[dr, "\pi"'] & (\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \arrow[d, "\alpha"] \\
& (Z_\star, m_{Z_\star}, \phi_\star)
\end{tikzcd}$$

gives that there is some R_ϕ-invariant family of quotients of Abelian groups $q_\star : A_\star \to A_{0,\star}$ such that

$$\begin{tikzcd}
\mathbf{X} \arrow[rr, leftrightarrow, "\cong"] \arrow[d, "\tilde{\pi}"'] & & (Z_\star, m_{Z_\star}, \phi_\star) \ltimes (A_\star, m_{A_\star}, \tau) \arrow[d, "\mathrm{id}_{Z_\star} \ltimes q_\star"] \\
(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\alpha"'] & & (Z_\star, m_{Z_\star}, \phi_\star) \ltimes (A_{0,\star}, m_{A_{0,\star}}, q_\star \circ \tau) \arrow[dl, "\text{canonical}"] \\
& (Z_\star, m_{Z_\star}, \phi_\star). &
\end{tikzcd}$$

Choosing a R_ϕ-invariant measurable selector $\eta_\star : A_{0,\star} \to A_\star$, we can now give an explicit re-coordinatization of the extension $\tilde{\pi} : \mathbf{X} \to (\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star)$ as

$$\begin{tikzcd}
\mathbf{X} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\tilde{\pi}"'] & (Z_\star \ltimes A_{0,\star}, m_{Z_\star \ltimes A_{0,\star}}, (\phi_\star \ltimes \lambda_\star)) \ltimes (\ker q_\star, m_{\ker q_\star}, \tilde{\tau}) \arrow[d, "\text{canonical}"] \\
(\tilde{Z}_\star, m_{\tilde{Z}_\star}, \tilde{\phi}_\star) \arrow[r, leftrightarrow, "\cong"] & (Z_\star \ltimes A_{0,\star}, m_{Z_\star \ltimes A_{0,\star}}, (\phi_\star \ltimes \lambda_\star))
\end{tikzcd}$$

for a suitable measurable selection of dense homomorphisms $\lambda_\star : \mathbb{Z}^2 \longrightarrow A_{0,\star}$, where the top isomorphism is obtained by composing the previous coordinatization $\mathbf{X} \cong (Z_\star, m_{Z_\star}, \phi_\star) \ltimes (A_\star, m_{A_\star}, \tau)$ with the map

$$((s,z),a) \mapsto ((s,z),\ q_s(a),\ a \cdot \eta_s(q_s(a))^{-1}).$$

This results in a cocycle

$$\tilde{\tau}(\mathbf{n},(s,z,a_0)) := \tau(\mathbf{n},(s,z)) \cdot \big(\eta_s(a_0 \cdot q_s(\tau(\mathbf{n},(s,z)))) \cdot \eta_s(a_0)^{-1}\big)^{-1} \in \ker q_s$$

for $(s,z,a_0) \in S \ltimes Z_\star \ltimes A_{0,\star}$.

As in Step 1, it remains simply to verify that for any measurably-varying $\chi_\star \in \widehat{\ker q_\star}$ the cocycle $\tilde{\tau} : Z_\star \ltimes A_{0,\star} \to \ker q_\star$ admits S^1-valued solutions to the equations

$$\mathrm{E}(u_\star, \phi_\star(\mathbf{n}_1), \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}, \chi_\star \circ \tilde{\tau}(\mathbf{n},\,\cdot\,))$$

for every $\mathbf{n} \in \Gamma$ and $u_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}$, and

$$\mathrm{E}(v_\star, \phi_\star(\mathbf{n}_1), \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}, \chi_\star \circ \tilde{\tau}(\mathbf{n},\,\cdot\,))$$

for every $\mathbf{n} \in \Gamma$ and $v_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_3)}$. We will treat the first of these, the second being exactly similar. Suppose that $\mathbf{n} \in \Gamma$, that $\chi_\star \in \widehat{\ker q_\star}$ which we arbitrarily

extend to a measurable selection from $\widehat{A_\star}$, that $u_\star \in \overline{\phi_\star(\mathbb{Z}\mathbf{n}_2)}$ and that $b_\star$ is a solution to the corresponding equation:

$$\Delta_{u_s}(\chi_s \circ \tau)(\mathbf{n}, z) = \Delta_{\phi_s(\mathbf{n})} b_s(z) \cdot c_s(z \cdot \overline{\phi_s(\mathbb{Z}\mathbf{n}_3)}) \qquad \text{for } m_{Z_s}\text{-a.e. } z \in Z_s$$

for ν-a.e. $s \in S$. Let $\tilde{u}_\star$ be any measurable lift of $u_\star$ through α to a measurable selection from $\overline{\tilde{\phi}_s(\mathbb{Z}\mathbf{n}_2)} \leq \tilde{Z}_s$. Then from the definition of $\tilde{\tau}$ we have

$$\Delta_{\tilde{u}_s}(\chi_s \circ \tilde{\tau})(\mathbf{n}, \tilde{z}) = \Delta_{u_s}(\chi_s \circ \tau)(\mathbf{n}, z) \cdot \Delta_{\tilde{u}_s} \Delta_{\tilde{\phi}_s(\mathbf{n})} b'_s(\tilde{z})$$

where $b'_s(\tilde{z})$ is the function $\tilde{Z}_s \to \mathrm{S}^1$ that corresponds to the function

$$Z_s \ltimes A_{0,s} \to \mathrm{S}^1 : (z, a_0) \mapsto \chi_s(\eta_s(a_0))^{-1}$$

under the above isomorphism $\tilde{Z}_s \leftrightarrow Z_s \ltimes A_{0,s}$, simply because under this isomorphism the expression $q_s(\tau(\mathbf{n}, (s, z)))$ appearing in the definition of $\tilde{\tau}$ describes the lift of the rotation by $\phi_s(\mathbf{n}) \in Z_s$ to the rotation by $\tilde{\phi}_s(\mathbf{n}) \in \tilde{Z}_s$.

Hence adjusting $b_\star$ to $\tilde{b}_\star : (s, \tilde{z}) \mapsto b_s(\alpha(\tilde{z})) \cdot \Delta_{\tilde{u}_s} b'_s(\tilde{z})$ and letting $\tilde{c}_s(\tilde{z}) := c_s(\alpha(\tilde{z}))$ we obtain a solution to the equation $\mathrm{E}(u_\star, \tilde{\phi}_\star(\mathbf{n}), \overline{\tilde{\phi}_\star(\mathbb{Z}\mathbf{n}_3)}, \chi_\star \circ \tilde{\tau}(\mathbf{n}, \cdot\,))$ over the lifted system, as required. This completes the proof. $\square$

Remark We make the assumption that $\tilde{\pi}$ is relatively ergodic because if we start with a non-ergodic directional CL-extension $\mathbf{X} \to (Z_\star, m_{Z_\star}, \phi_\star)$ then it will also admit many intermediate systems that are relatively invariant over $(Z_\star, m_{Z_\star}, \phi_\star)$ and are given by some complicated combination of cosets of the Mackey group. $\lhd$

Corollary 3.9. *Any joining of two $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-systems is a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-system.*

Proof By the preceding proposition we may regard two directional CL-systems as directional CL-extensions of their Kronecker factors (that is, their maximal factors that are expressible as direct integrals of ergodic group rotations). Now as explained previously the joining of those is another direct integral of ergodic group rotations, and over this the overall joining is simply given as an Abelian group extension with measure supported by some cosets of the Mackey group data inside the product of the fibre data of the two original systems. Even if this Abelian extension is not relatively ergodic, we can still multiply solutions to the individual directional CL-equations to show that the directional CL-equations for the combined cocycle also always admit solutions, as required (once again, this is possible because we define directional CL-cocycles by considering only their image under the fibrewise application of an arbitrary measurable selection of fibre group characters). $\square$

Proposition 3.8 also enables us to take inverse limits of directional CL-systems.

Corollary 3.10. *Any inverse limit of* $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$*-directional CL-systems is a* $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$*-directional CL-system.*

Proof After using Proposition 3.8 to write each of our contributing directional CL-systems as a directional CL-extension of its Kronecker factor, this now follows from the Relative Factor Structure Theorem by first adjoining the Kronecker factor of the inverse limit to each individual system in the sequence to give a new sequence expressed as an inverse limit of directional CL-extensions of the same base Kronecker system, and then applying the third part of Lemma 3.7. $\square$

The following is also an immediate consequence of the above definition and results.

Lemma 3.11. *If* $\mathbf{X}$ *is a* $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$*-directional CL-system then so are almost all of its ergodic components.*

Proof Indeed, upon expressing the system as $(Z_\star, m_{Z_\star}, \phi_\star) \ltimes (A_\star, m_{A_\star}, \sigma)$ so that the invariant base space S of this direct integral coordinatizes the whole of the invariant factor, almost every ergodic component is of the form $(Z_s, m_{Z_s}, \phi_s) \ltimes (A_s, m_{A_s}, \sigma)$ and so is manifestly also a $(\Gamma, \mathbf{n}_2, \mathbf{n}_3)$-directional CL-system. $\square$

With this in hand we can now prove the following useful addendum to Theorem 1.2.

Lemma 3.12. *If* $\mathbf{X}$ *is ergodic, then the pleasant extension* $\pi : \tilde{\mathbf{X}} \to \mathbf{X}$ *output by Theorem 1.2 may also be assumed to be ergodic.*

For the introduction of satedness and the definition of an FIS system, see Subsection 3.1 of [4].

Proof First we note that by alternately implementing Theorem 1.2 and constructing an FIS extension and then taking an inverse limit, we may always assume that the system output by that Theorem is FIS.

Now given an extension $\pi : \tilde{\mathbf{X}} \to \mathbf{X}$, if $\mathbf{X}$ is ergodic then almost every ergodic component of $\tilde{\mu}$ must still push down onto μ under π, so almost every ergodic component of $\tilde{\mathbf{X}}$ still defines an extension of $\mathbf{X}$. Let us write $\tilde{\mu}_\omega$, $\omega \in \Omega$, for some standard Borel parameterization of the ergodic components of $\tilde{\mu}$.

We next show that if $\xi_i : \mathbf{X} \to \mathbf{Y}_i$ are the characteristic factors of the original system and $\tilde{\xi}_i$ is the join of isotropy and directional CL-systems appearing in the characteristic triple for the system $\tilde{\mathbf{X}}$, then $\tilde{\xi}_i$ must still contain ξ_i for almost every $\tilde{\mu}_\omega$. Let $(A_m)_{m\geq 1}$ be a sequence of ξ_i-measurable subsets of X that generate the whole ξ_i-measurable σ-algebra up to μ-negligible sets. Since almost every $\tilde{\mu}_\omega$ is still a lift of μ under π, it follows that $(\pi^{-1}(A_m))_{m\geq 1}$ still generates the whole

$(\xi_i \circ \pi)$-measurable σ-algebra up to $\tilde{\mu}_\omega$-negligible sets for almost every $\tilde{\mu}_\omega$. On the other hand, since $\xi_i \precsim \tilde{\xi}_i$ for $\tilde{\mu}$, we know that there are corresponding $\tilde{\xi}_i$-measurable subsets $B_m \subseteq \tilde{X}$ such that $\tilde{\mu}(\pi^{-1}(A_m)\triangle B_m) = 0$ for all $m \geq 1$. This must now also still hold for almost every $\tilde{\mu}_\omega$, and so we have deduced that under almost every $\tilde{\mu}_\omega$ the σ-algebra generated by $\tilde{\xi}_i$ contains that generated by $\xi_i \circ \pi$ up to negligible sets.

Finally, we observe that $(\tilde{\xi}_i)_\# \tilde{\mu}$ is a joining of three isotropy systems and a directional CL-system, and so by the previous lemma and its obvious analog for isotropy systems we deduce that $\tilde{\xi}_i$ is also a joining of (ergodic) isotropy systems and a directional CL-system for almost every $\tilde{\mu}_\omega$.

Thus we have shown that any ergodic $\mathbf{X}$ admits an ergodic extension $(\tilde{X}, \tilde{\mu}_\omega, \tilde{T})$ such that the characteristic triple of factors in $\mathbf{X}$ is still determined by the corresponding joins of systems given by Theorem 1.2. It is less clear that the lifted characteristic factors $\tilde{\xi}_i$ are still generated by isotropy and directional CL-systems up to negligible sets for almost every $\tilde{\mu}_\omega$, but this problem can be easily repaired by iterating this construction and then taking the (still ergodic) inverse limit of the tower of extensions that results. $\square$

By taking ergodic decompositions, it is clear that the norm convergence asserted by Theorem 1.1 holds in general if and only if it holds for every ergodic $\mathbb{Z}^2$-action, and given this observation and the above lemma we will now restrict our attention to ergodic systems for the rest of the paper.

3.2 First reduction

We now return to the consideration of the averages $S_N(\cdot,\cdot)$. Our first simplification will follow from Theorem 1.3, giving an identification of a pair of characteristic factors in a pleasant extension for our quadratic averages of interest. Having obtained this, by manipulating the classes of functions that result we will see how to simplify the averages we need to consider even further.

Theorem. *Any ergodic $\mathbb{Z}^2$-system $\mathbf{X}_0$ admits an ergodic extension $\pi : \mathbf{X} \to \mathbf{X}_0$ in which some factor*

$$\xi_1 = \xi_2 := \zeta_{\mathrm{pro}}^{T^{\mathbf{e}_1}} \vee \zeta_0^{T^{\mathbf{e}_2}} \vee \bigvee_{h\geq 1} \eta_h$$

is characteristic for the averages $S_N(\cdot,\cdot)$, where each η_h is a factor of $\mathbf{X}$ whose target is an $(h\mathbb{Z}^2, h\mathbf{e}_1, h\mathbf{e}_2)$-directional CL-system for the lattice $h\mathbb{Z}^2 := \{(hm,hn) :$

$m,n \in \mathbb{Z}\}$, and so

$$S_N(f_1,f_2) \sim S_N(\mathsf{E}_\mu(f_1\,|\,\xi_1),\mathsf{E}_\mu(f_2\,|\,\xi_2))$$

in $L^2(\mu)$ as $N \to \infty$ for any $f_1,f_2 \in L^\infty(\mu)$.

We will prove this in a number of steps.

Lemma 3.13. *If*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1\circ T_1^{n^2})(f_2\circ T_1^{n^2}T_2^n) \not\to 0$$

in $L^2(\mu)$ as $N \to \infty$ then there are some $\varepsilon > 0$ and an increasing sequence of integers $1 \le h_1 < h_2 < \ldots$ such that

$$\Big\|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(f_1\circ T_1^{h_i^2}\circ T_1^{2h_in})(f_2\circ T_2^n)(f_2\circ(T_1^{h_i^2}T_2^{h_i})\circ(T_1^{2h_i}T_2)^n)\Big\|_2^2 \ge \varepsilon$$

for each $i \ge 1$.

Proof Setting $u_n := (f_1\circ T_1^{n^2})(f_2\circ T_1^{n^2}T_2^n) \in L^2(\mu)$, the version of the classical van der Corput estimate for bounded Hilbert space sequences (see, for instance, Section 1 of Furstenberg and Weiss [13]) shows that

$$\frac{1}{N}\sum_{n=1}^{N}(f_1\circ T_1^{n^2})(f_2\circ T_1^{n^2}T_2^n) \not\to 0$$

in $L^2(\mu)$ as $N \to \infty$ only if

$$\begin{aligned}&\frac{1}{H}\sum_{h=1}^{H}\frac{1}{N}\sum_{n=1}^{N}\langle u_n,u_{n+1}\rangle\\ &=\frac{1}{H}\sum_{h=1}^{H}\int_X f_1\cdot\frac{1}{N}\sum_{n=1}^{N}((f_1\circ T_1^{h^2})\circ T_1^{2hn})(f_2\circ T_2^n)((f_2\circ T_1^{h^2}T_2^h)\circ T_1^{2hn}T_2^n)\,\mathrm{d}\mu\\ &\not\to 0,\end{aligned}$$

and hence, by the Cauchy-Schwartz inequality, only if $f_1 \neq 0$ and for some $\varepsilon > 0$

there is an increasing sequence $1 \leq h_2 < h_2 < \dots$ such that

$$\|f_1\|_2^2 \Big\| \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N (f_1 \circ T_1^{h_i^2} \circ T_1^{2h_i n})(f_2 \circ T_2^n)(f_2 \circ (T_1^{h_i^2} T_2^{h_i}) \circ (T_1^{2h_i} T_2)^n) \Big\|_2^2$$
$$\geq \Big| \int_X f_1 \cdot \Big(\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N (f_1 \circ T_1^{h_i^2} \circ T_1^{2h_i n})(f_2 \circ T_2^n)(f_2 \circ (T_1^{h_i^2} T_2^{h_i}) \circ (T_1^{2h_i} T_2)^n) \Big) \,\mathrm{d}\mu \Big|$$
$$\geq \|f_1\|_2^2 \varepsilon$$

as required. □

In view of Theorem 1.2 and a judicious appeal to Lemma 3.2 this immediately implies the following.

Corollary 3.14. *Any ergodic $\mathbb{Z}^2$-system $\mathbf{X}_0$ admits an ergodic extension $\pi : \mathbf{X} \to \mathbf{X}_0$ such that if $S_N(f_1, f_2) \not\to 0$ in $L^2(\mu)$ as $N \to \infty$ for some $f_1, f_2 \in L^\infty(\mu)$ then there are some $\varepsilon > 0$ and an increasing sequence of integers $1 \leq h_1 < h_2 < \dots$ such that*

$$\big\| \mathsf{E}_\mu(f_1 \,|\, \zeta_0^{T_1^{2h_i}} \vee \zeta_0^{T_1^{2h_i} T_2^{-1}} \vee \zeta_0^{T_2^{-1}} \vee \eta_{1,h_i}) \big\|_2^2 \geq \varepsilon$$

and

$$\big\| \mathsf{E}_\mu(f_2 \,|\, \zeta_0^{T_1^{2h_i}} \vee \zeta_0^{T_1^{2h_i} T_2} \vee \zeta_0^{T_2} \vee \eta_{2,h_i}) \big\|_2^2 \geq \varepsilon$$

for each $i \geq 1$, where each η_{1,h_i} is a factor of $\mathbf{X}$ whose target is a $((2h_i, -1), (2h_i, 0), (0, -1))$-directional CL-system and each η_{2,h_i} is a factor whose target is a $((2h_i, 1), (2h_i, 0), (0, 1))$-directional CL-system (noting that for these triples of directions all of the values m_{ij} appearing in Theorem 1.2 equal ± 1). □

This corollary tells us that if $S_N(f_1, f_2) \not\to 0$ then each of f_1 and f_2 must enjoy a large conditional expectation onto not just one factor of $\mathbf{X}$ with a special structure, but a whole infinite sequence of these factors. We will now use this to cut down the characteristic factors we need for the averages S_N further by examining the possible joint distributions of the members of these infinite families of factors. For this we need to recall the following special property of certain Kronecker systems, introduced in Subsection 4.8 of [5].

Definition 3.15 (DIO system)**.** *A $\mathbb{Z}^d$-Kronecker system (Z, m_Z, ϕ), where $\phi : \mathbb{Z}^d \longrightarrow Z$ is a homomorphism, has the* ***disjointness of independent orbits property*** *or is* ***DIO*** *if for any subgroups $\Gamma_1, \Gamma_2 \leq \mathbb{Z}^2$ we have*

$$\Gamma_1 \cap \Gamma_2 = \{\mathbf{0}\} \qquad \Rightarrow \qquad \overline{\phi(\Gamma_1)} \cap \overline{\phi(\Gamma_2)} = \{1_Z\}.$$

The following was Proposition 4.32 in [5]:

Lemma 3.16. *If a $\mathbb{Z}^2$-system is FIS then its Kronecker factor is DIO, and consequently any $\mathbb{Z}^2$-Kronecker system has a Kronecker extension that is DIO.* $\square$

We will also need the following base result on factorizing transfer functions, which appears as Lemma 10.3 in Furstenberg and Weiss [13].

Lemma 3.17. *If $\mathbf{X}_i$ for $i = 1, 2$ are ergodic $\mathbb{Z}$-systems and $f_i : X_i \to \mathrm{S}^1$ are Borel maps for which there is some Borel $g : X_1 \times X_2 \to \mathrm{S}^1$ with $f_1 \otimes f_2 = \Delta_{T_1 \times T_2} g$, $(\mu_1 \otimes \mu_2)$-a.s., then in fact there are constants $c_i \in \mathrm{S}^1$ and Borel maps $g_i : X_i \to \mathrm{S}^1$ such that $f_i = c_i \cdot \Delta_{T_i} g_i$.* $\square$

Lemma 3.18. *Suppose that $h_1 \neq h_2$ are distinct nonzero integers and let $h := \mathrm{l.c.m.}(h_1, h_2, h_1 - h_2, h_1 + h_2)$. Suppose that $\mathbf{X}$ is an ergodic $\mathbb{Z}^2$-system with a pair of factors*

$$\begin{tikzcd} & \mathbf{X} \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] \arrow[dd, "\zeta_1^T"] & \\ \mathbf{Y}_1 \arrow[dr, "\zeta_1"'] & & \mathbf{Y}_2 \arrow[dl, "\zeta_2"] \\ & \mathbf{Z}_1^T & \end{tikzcd}$$

such that each η_i is an $((h_i, 1), (h_i, 0), (0, 1))$-directional CL-extension of ζ_i, and that the Kronecker system $\mathbf{Z}_1^T$ is DIO. Then η_1 and η_2 are relatively independent under μ over some further common factor $\eta : \mathbf{X} \to \mathbf{Y}$ located as in the diagram

$$\begin{tikzcd} & \mathbf{X} \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] \arrow[dd, "\eta"] & \\ \mathbf{Y}_1 \arrow[dr, "\alpha_1"'] & & \mathbf{Y}_2 \arrow[dl, "\alpha_2"] \\ & \mathbf{Y} \arrow[d, "\zeta_1^T|_\eta"] & \\ & \mathbf{Z}_1^T, & \end{tikzcd}$$

and where $\mathbf{Y}$ is an $(h\mathbb{Z}^2, (h, 0), (0, h))$-directional CL-system.

Proof For $i = 1, 2$ let us pick a coordinatization

$$\begin{tikzcd}
\mathbf{Y}_i \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\zeta_i"'] & (Z, m_Z, \phi) \ltimes (A_i, m_{A_i}, \sigma_i) \arrow[d, "\text{canonical}"] \\
\mathbf{Z}_1^T \arrow[r, leftrightarrow, "\cong"] & (Z, m_Z, \phi),
\end{tikzcd}$$

so σ_i is an $((h_i, 1), (h_i, 0), (0, 1))$-directional CL-cocycle over R_ϕ.

These now combine to give a coordinatization of the target system of the joint factor $\eta_1 \vee \eta_2$ of $\mathbf{X}$ as an extension of $\mathbf{Z}_1^T \cong (Z, m_Z, \phi)$ by some $(R_\phi \ltimes (\sigma_1, \sigma_2))$-invariant lift of m_Z to the space $Z \ltimes (A_1 \times A_2)$. Calling this invariant lifted measure ν, we know that its two coordinate projections onto $Z \ltimes A_i$ must be simply $m_Z \ltimes m_{A_i}$ (since this is just the measure on the system $\mathbf{Z}_i$), and that it is relatively ergodic for the $\mathbb{Z}^2$-action $R_\phi \ltimes (\sigma_1, \sigma_2)$ over the canonical factor map onto (Z, m_Z, ϕ), simply because the whole of $\mathbf{X}$ is ergodic.

Therefore it follows from the Mackey Theorem describing ergodic components of isometric extensions (see Proposition 4.7 in [3]) that ν takes the form $m_Z \ltimes m_{b(\bullet)^{-1}M}$ for some section $b : Z \to A_1 \times A_2$ and some Mackey group $M \leq A_1 \times A_2$ that has full one-dimensional projections onto A_1 and A_2.

Now, in this description of ν we are free to alter b pointwise by any M-valued section, and so since M has full one-dimensional projections we may assume without loss of generality that b takes values in $\{1_{A_1}\} \times A_2$. Now simply identifying $\{1_{A_1}\} \times A_2$ with a copy of the group A_2, if we adjust our above coordinatization of the extension $\mathbf{Y}_i \xrightarrow{\zeta_i} \mathbf{Z}_1^T$ by fibrewise rotation by $b(\bullet)^{-1}$ we obtain a new coordinatization of this extension by a compact Abelian group and cocycle with all the properties of our initially-chosen coordinatization, and such that the resulting Mackey data of the combined coordinatization has $b \equiv 1_{A_1 \times A_2}$.

Re-assigning our initial notation to this new coordinatization, we now have $\nu = m_Z \ltimes m_M$ for some fixed $M \leq A_1 \times A_2$. It follows that the two coordinate-projection factors of the joined system $(Z \ltimes (A_1 \times A_2), \nu, R_\phi \ltimes (\sigma_1, \sigma_2))$ onto $Z \ltimes A_i$ are relatively independent over their further factors given by the maps

$$Z \ltimes A_i \to Z \ltimes (A_i/M_i) : (z, a) \mapsto (z, aM_i)$$

where M_i for $i = 1, 2$ are the one-dimensional slices of the Mackey group M. Moreover, the targets of these two factor maps are identified within $(Z \ltimes (A_1 \times A_2), \nu, R_\phi \ltimes (\sigma_1, \sigma_2))$ (and hence within $\mathbf{X}$), because $M/(M_1 \times M_2)$ is now a

subgroup of $(A_1/M_1) \times (A_2/M_2)$ that has full one-dimensional projections and trivial slices, and therefore defines the graph of an isomorphism. This common target therefore specifies some common Abelian subextension $\eta_1, \eta_2 \succsim \eta \succsim \zeta_1^T$ over which the η_i are relatively independent.

This identifies the factor η promised by the proposition; it remains to show that its target is an $(h\mathbb{Z}^2, (h,0), (0,h))$-directional CL-system.

First let $A \cong A_1/M_1 \cong A_2/M_2$ be the fibre group of some coordinatization of η over ζ_1^T, $q_i : A_i \twoheadrightarrow A$ a continuous epimorphism that corresponds to quotienting by the subgroup M_i, and $\sigma : \mathbb{Z}^2 \times Z \to A$ the cocycle over R_ϕ of this coordinatization (so $\sigma = q_i \circ \sigma_i$ for $i = 1, 2$). Now let $\chi \in \widehat{A}$, and let $\chi_i := \chi \circ q_i \in \widehat{A_i}$ for $i = 1, 2$.

For any $u \in \overline{\phi(\mathbb{Z} \cdot (0,1))}$ the equation $\mathrm{E}(u, \phi(h_i, 1), \overline{\phi(\mathbb{Z} \cdot (h_i, 0))}, \chi_i \circ \sigma_i((h_i, 1),\,\cdot\,))$ gives a solution $b_{i,u} : Z \to \mathrm{S}^1$ together with a one-dimensional auxiliary $c_{i,u} : Z/\overline{\phi(\mathbb{Z} \cdot (h_i, 0))} \to \mathrm{S}^1$ such that

$$\Delta_u \chi_i(\sigma_i((h_i, 1), z)) = \Delta_{\phi(h_i,1)} b_{i,u}(z) \cdot c_{i,u}(z \cdot \overline{\phi(\mathbb{Z} \cdot (h_i, 0))}),$$

and hence in fact

$$\Delta_u \chi(\sigma((h_i, 1), z)) = \Delta_{\phi(h_i,1)} b_{i,u}(z) \cdot c_{i,u}(z \cdot \overline{\phi(\mathbb{Z} \cdot (h_i, 0))})$$

for $i = 1, 2$, because $\chi \circ \sigma = \chi \circ q_i \circ \sigma_i = \chi_i \circ \sigma_i$. We will show that by modifying $b_{i,u}$ for either $i = 1$ or $i = 2$ we can produce a map that simultaneously satisfies the equations $\mathrm{E}(u, \phi(\mathbf{n}), \overline{\phi(\mathbb{Z} \cdot (h, 0))}, \chi \circ \sigma(\mathbf{n},\,\cdot\,))$ for all $\mathbf{n} \in h\mathbb{Z}^2$. Since the case of any $v \in \overline{\phi(\mathbb{Z} \cdot (h_1, 0))} \cap \overline{\phi(\mathbb{Z} \cdot (h_2, 0))} \supseteq \overline{\phi(\mathbb{Z} \cdot (h, 0))}$ is symmetrical, this will complete the proof.

We can apply the differencing operator $\Delta_{\phi(h,0)}$ to the above equation to obtain

$$\Delta_u \Delta_{\phi(h,0)} \chi(\sigma((h_i, 1), z)) = \Delta_{\phi(h_i,1)} \Delta_{\phi(h,0)} b_{i,u}(z),$$

where we have used the commutativity of differencing and the fact that $(h, 0) \in \mathbb{Z} \cdot (h_i, 0)$ and so

$$\Delta_{\phi(h,0)} c_{i,u}(z \cdot \overline{\phi(\mathbb{Z} \cdot (h_i, 0))}) \equiv 1.$$

On the other hand, we can now appeal to the cocycle equation $\Delta_{\phi(h,0)} \chi(\sigma((h_i, 1), z)) = \Delta_{\phi(h_i,1)} \chi(\sigma((h, 0), z))$ to re-write the above as

$$\Delta_{\phi(h_i,1)} \big(\Delta_u \chi(\sigma((h, 0), z)) \cdot \Delta_{\phi(h,0)} b_{i,u}(z)^{-1}\big) \equiv 1,$$

and so we can write

$$\Delta_u \chi(\sigma((h, 0), z)) \cdot \Delta_{\phi(h,0)} b_{i,u}(z)^{-1} = f_{i,u}(z \cdot \overline{\phi(\mathbb{Z} \cdot (h_i, 1))}),$$

for some $f_{i,u} : Z/\overline{\phi(\mathbb{Z}\cdot(h_i,1))} \to \mathrm{S}^1$.

Finally, taking the difference of these last equations for $i = 1$ and for $i = 2$ we find

$$\Delta_{\phi(h,0)}(b_{2,u} \cdot b_{1,u}^{-1}) = (f_{1,u} \circ r_1) \cdot \overline{(f_{2,u} \circ r_2)}$$

where r_i is the quotient epimorphism $Z \to Z/\overline{\phi(\mathbb{Z}\cdot(h_i,1))}$.

Now, on the one hand $(h,0) \in \mathbb{Z}\cdot(h_1,1) + \mathbb{Z}\cdot(h_2,1)$, and on the other we know that $\overline{\phi(\mathbb{Z}\cdot(h_1,1))} \cap \overline{\phi(\mathbb{Z}\cdot(h_2,1))} = \{1_Z\}$ by the DIO assumption. Therefore we can analyze the above equation by applying Lemma 3.17 for each pair of ergodic components of the restrictions $R_{\phi(h,0)}|_{r_i}$, $i = 1, 2$, since the disjointness of the two orbit-closures tells us that the above equation restricts to a combined coboundary equation simply on the direct product of those two ergodic components. This tells us that in fact the function $f_{i,u}(r_i(z))$ must take the form

$$\Delta_{\phi(h,0)}(b'_{i,u} \circ r_i(z)) \cdot g_{i,u}(z \cdot \overline{\phi(\Gamma')})$$

for some Borel maps $b'_{i,u} : Z/\overline{\phi(\mathbb{Z}\cdot(h_i,1))} \to \mathrm{S}^1$ and $g_{i,u} : Z/\overline{\phi(\Gamma')} \to \mathrm{S}^1$ $\Gamma' := \mathbb{Z}\cdot(h_1,1) + \mathbb{Z}\cdot(h,0)$. Since $\Gamma' \supseteq h\mathbb{Z}^2$, we may instead regard $g_{i,u}$ as a map $Z/\overline{\phi(h\mathbb{Z}^2)} \to \mathrm{S}^1$ and write the above function as

$$\Delta_{\phi(h,0)}(b'_{i,u} \circ r_i(z)) \cdot g_{i,u}(z \cdot \overline{\phi(h\mathbb{Z}^2)}).$$

It also follows easily from the Measurable Selector Theorem that we can take the above equations to hold for Haar-a.e. u using Borel selections $u \mapsto b_{i,u}, g_{i,u}$.

Now, clearly $b'_{i,u} \circ r_i$ is invariant under $R_{\phi(h_i,1)}$, and so $\Delta_{\phi(h_i,1)}(b_{i,u} \cdot (b'_{i,u} \circ r_i)) = \Delta_{\phi(h_i,1)} b_{i,u}$. This means we can simply replace $b_{i,u}$ with $(b_{i,u} \cdot (b'_{i,u} \circ r_i))$ in our original directional Conze-Lesigne equation, and hence assume that the solutions we obtained for that equation also satisfy

$$\Delta_u \chi(\sigma((h,0),z)) \cdot \Delta_{\phi(h,0)} b_{i,u}(z)^{-1} = g_{i,u}(z \cdot \overline{\phi(h\mathbb{Z}^2)}).$$

However, this now re-arranges into the form

$$\Delta_u \chi(\sigma((h,0),z)) = \Delta_{\phi(h,0)} b_{i,u}(z) \cdot g_{i,u}(z \cdot \overline{\phi(h\mathbb{Z}^2)}),$$

and so since $\mathbb{Z}\cdot(0,h) \subseteq h\mathbb{Z}^2$ and $\mathbb{Z}\cdot(0,h) \subseteq \mathbb{Z}\cdot(0,h_i)$, this new version of $b_{i,u}$ is a solution to both the originally-assumed equation

$$\mathrm{E}(u, \phi(h_i,1), \overline{\phi(\mathbb{Z}\cdot(h,0))}, \chi \circ \sigma((h_i,1),\cdot))$$

and also the equation

$$\mathrm{E}(u,\phi(h,0),\overline{\phi(\mathbb{Z}\cdot(h,0))},\chi\circ\sigma((h,0),\cdot))$$

(with different one-dimensional auxiliaries), and so is actually a solution to

$$\mathrm{E}(u,\phi(\mathbf{n}),\overline{\phi(\mathbb{Z}\cdot(h,0))},\chi\circ\sigma(\mathbf{n},\cdot))$$

for every $\mathbf{n}\in\mathbb{Z}\cdot(h,0)+\mathbb{Z}\cdot(h_i,1)\supseteq h\mathbb{Z}^2$, as required. □

We will shortly use the above lemma to examine the joint distributions of the families of characteristic factors obtained from Corollary 3.14. However, before doing so we record the following corollary of the above proof, which will be useful later.

Corollary 3.19. *If $\sigma:\mathbb{Z}^2\times Z\to\mathrm{S}^1$ is a $(\Gamma,\mathbf{n}_2,\mathbf{n}_3)$-directional CL-cocycle over a DIO system where $\mathbf{n}_2,\mathbf{n}_3\in\Gamma$, then for $\{i,j\}=\{2,3\}$ there are Borel maps $b_i:K_{\mathbf{n}_i}\times Z\to\mathrm{S}^1$ and $c:K_{\mathbf{n}_i}\times Z/\overline{\phi(\Gamma)}\to\mathrm{S}^1$ such that*

$$\Delta_u\sigma(\mathbf{n}_j,\cdot)=b_i(u,z\cdot\phi(\mathbf{n}_j))\cdot\overline{b_i(u,z)}\cdot c(u,z\cdot\overline{\phi(\Gamma)})$$

for Haar-almost every $(u,z)\in K_{\mathbf{n}_i}\times Z$.

Proof For a fixed $u\in K_{\mathbf{n}_i}$, the construction of the new function $g_{i,u}$ in the previous proof shows that we may find a solution together with a one-dimensional auxiliary c_u for the directional CL-equation $\mathrm{E}(u,\phi(\mathbf{n}_j),\overline{\phi(\Gamma)},\chi\circ\sigma(\mathbf{n}_j,\,\cdot\,))$ — in particular, such that $c_u(z)$ actually depends only on the coset $z\cdot\overline{\phi(\Gamma)}$.

It now follows from a simple measurable selection argument applied to the collection

$$\big\{(u,b',c')\in K_{\mathbf{n}_i}\times\mathcal{C}(K_{\mathbf{n}_i}\times Z)\times\mathcal{C}(K_{\mathbf{n}_i}\times Z/\overline{\phi(\Gamma)}):\\ \Delta_u\sigma(\mathbf{n}_j,\cdot)=b'(z\cdot\phi(\mathbf{n}_j))\cdot\overline{b'(z)}\cdot c'(z\cdot\overline{\phi(\Gamma)})\big\}$$

(where as usual $\mathcal{C}(U)$ denotes the Polish group of equivalence classes of Borel maps $U\to\mathrm{S}^1$ under m_U-a.e. agreement, endowed with the topology of convergence in probability) that we may take a selection of maps $b_{i,u}$ and $c_{i,u}$ that is Borel in u and satisfies this almost-sure equation for a.e. u.

It remains to obtain measurable functions b_i on $K_{\mathbf{n}_i}\times Z$ and c on $K_{\mathbf{n}_i}\times Z/\overline{\phi(\Gamma)}$ such that $b_i(u,z)=b_{i,u}(z)$ and $c_i(u,z\overline{\phi(\Gamma)})=c_{i,u}(z\overline{\phi(\Gamma)})$ for a.e. (u,z) and hence that satisfy the desired equation Haar-almost everywhere. This can be done, for example, by identifying (Z,m_Z) with $([0,1),\mathrm{Lebesgue})$ as standard Borel

probability spaces and then defining $b_i(u,z)$ as the pointwise limit of the (well-defined) averages of $b_{i,u}$ over increasingly short dyadic intervals of values of z. By the Lebesgue Density Theorem these averages converge almost everywhere, and the resulting pointwise limit function is clearly jointly measurable in (u,z) and agrees with $b_{i,u}$ almost surely for almost every u. A similar construction applies to $c_{i,u}$, and we can make these functions Borel by making one further modification on a negligible set. $\square$

The immediate application we have for Lemma 3.18 will require also some basic results on the possible distributions of collections of one-dimensional isotropy factors of a $\mathbb{Z}^2$-system.

Lemma 3.20. *Suppose that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3 \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ are three directions no two of which are parallel, that $\mathbf{X}_1 = (X_1,\mu_1,T_1) \in \mathsf{Z}_0^{\mathbf{n}_1}$, $\mathbf{X}_2 = (X_2,\mu_2,T_2) \in \mathsf{Z}_0^{\mathbf{n}_2}$, $\mathbf{X}_3 = (X_3,\mu_3,T_3) \in \mathsf{Z}_0^{\mathbf{n}_3}$ and that $\mathbf{Z} = (Z,\nu,S)$ is a group rotation $\mathbb{Z}^2$-system. Suppose further that $\mathbf{X} = (X,\mu,T)$ is a joining of these four systems through the factor maps $\xi_i : \mathbf{X} \to \mathbf{X}_i$, $i = 1,2,3$ and $\alpha : \mathbf{X} \to \mathbf{Z}$. Then $(\xi_1,\xi_2,\xi_3,\alpha)$ are relatively independent under μ over their further factors $(\zeta_1^{T_1}\circ\xi_1,\zeta_1^{T_2}\circ\xi_2,\zeta_1^{T_3}\circ\xi_3,\alpha)$.*

Proof We will prove that under $\mathbf{X}$ the factors ξ_1, ξ_2, ξ_3 and α are relatively independent over $\zeta_1^{T_1}\circ\xi_1$, ξ_2, ξ_3 and α; repeating this argument to handle ξ_2 and ξ_3 then gives the full result.

Letting $\mathbf{Y} = (\xi_3 \vee \alpha)(\mathbf{X})$ be the factor of $\mathbf{X}$ generated by ξ_3 (which is $T^{\mathbf{n}_3}$-invariant) and α (which is isometric for T, hence certainly for $T^{\mathbf{n}_3}$), we see that this is a $T^{\mathbf{n}_3}$-isometric system. This implies that its joining to any other system is relatively independent over the maximal $T^{\mathbf{n}_3}$-isometric factor of that other system.

On the other hand, ξ_1 and ξ_2 must be relatively independent over $\xi_1\wedge\xi_2$ under μ (simply by averaging with respect to $\mathbf{n}_2$), and that the subactions generated by both $\mathbf{n}_1$ and $\mathbf{n}_2$ are trivial on this meet, so $\xi_1\wedge\xi_2 \precsim \zeta_0^{T^{\mathbf{n}_1},T^{\mathbf{n}_2}}$, whose target system is a direct integral of finite group rotations factoring through the quotient $\mathbb{Z}^2/(\mathbb{Z}\mathbf{n}_1+\mathbb{Z}\mathbf{n}_2)$.

Since $\xi_1\vee\xi_2$ must be joined to $\xi_3\vee\alpha$ relatively independently over the maximal $T^{\mathbf{n}_3}$-isometric factor of $\xi_1\vee\xi_2$, it follows from the Furstenberg-Zimmer Structure Theorem (recalled as Theorem 2.4 in [5]) that $\xi_1\vee\xi_2$ is in particular joined to $\xi_3\vee\alpha$ relatively independently over the join of maximal isometric subextensions

$$(\zeta_{1/(\xi_1\wedge\xi_2)|_{\xi_1}}^{T_1^{\mathbf{n}_3}}\circ\xi_1)\vee(\zeta_{1/(\xi_1\wedge\xi_2)|_{\xi_2}}^{T_2^{\mathbf{n}_3}}\circ\xi_2).$$

Since $\xi_1\wedge\xi_2$ has target a direct integral of *periodic* rotations, the maximal $T_i^{\mathbf{n}_3}$-

isometric subextension of $\xi_i \to (\xi_1 \wedge \xi_2)|_{\xi_i}$ is simply the maximal factor of ξ_i that is coordinatizable as a direct integral of group rotations for each $i = 1, 2$: that is, it is $\zeta_1^{T_i} \circ \xi_i$. Hence we have shown that under μ the factors $\xi_1 \vee \xi_2$ and $\xi_3 \vee \alpha$ are relatively independent over $(\zeta_1^{T_1} \circ \xi_1) \vee (\zeta_1^{T_2} \circ \xi_2)$ and $\xi_3 \vee \alpha$. Thus whenever $f_i \in L^\infty(\mu_i)$ for $i = 1, 2, 3$ and $g \in L^\infty(\nu)$ we have

$$
\begin{aligned}
&\int_X (f_1 \circ \xi_1) \cdot (f_2 \circ \xi_2) \cdot (f_3 \circ \xi_3) \cdot (g \circ \alpha)\,\mathrm{d}\mu \\
&= \int_X \mathsf{E}_\mu\big((f_1 \circ \xi_1) \cdot (f_2 \circ \xi_2)\,\big|\,(\zeta_1^{T_1} \circ \xi_1) \vee (\zeta_1^{T_2} \circ \xi_2)\big) \cdot (f_3 \circ \xi_3) \cdot (g \circ \alpha)\,\mathrm{d}\mu \\
&= \int_X (\mathsf{E}_\mu(f_1 \,|\, \zeta_1^{T_1}) \circ \xi_1) \cdot (\mathsf{E}_\mu(f_2 \,|\, \zeta_1^{T_2}) \circ \xi_2) \cdot (f_3 \circ \xi_3) \cdot (g \circ \alpha)\,\mathrm{d}\mu \\
&= \int_X (\mathsf{E}_\mu(f_1 \,|\, \zeta_1^{T_1}) \circ \xi_1) \cdot (f_2 \circ \xi_2) \cdot (f_3 \circ \xi_3) \cdot (g \circ \alpha)\,\mathrm{d}\mu,
\end{aligned}
$$

where the second equality follows from the relative independence of ξ_1 and ξ_2 over $\xi_1 \wedge \xi_2$, which is contained in $\zeta_1^{T_i} \circ \xi_i$ for both $i = 1, 2$. This completes the proof. $\square$

Lemma 3.21. *Suppose that $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4 \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ are directions no two of which are parallel, that $\mathbf{X}_i = (X_i, \mu_i, T_i) \in \mathsf{Z}_0^{\mathbf{n}_i}$ for $i = 1, 2, 3, 4$ and that $\mathbf{Y} = (Y, \nu, S)$ is a two-step Abelian isometric $\mathbb{Z}^2$-system. Suppose further that $\mathbf{X} = (X, \mu, T)$ is a joining of these five systems through the factor maps $\xi_i : \mathbf{X} \to \mathbf{X}_i$, $i = 1, 2, 3, 4$ and $\eta : \mathbf{X} \to \mathbf{Y}$, with the maximality properties that $\xi_i = \zeta_0^{T^{\mathbf{n}_i}}$ for $i = 1, 2, 3, 4$ and $\eta \gtrsim \zeta_1^T$. Then $(\xi_1, \xi_2, \xi_3, \xi_4, \eta)$ are relatively independent under μ over their further factors $(\zeta_{\mathrm{Ab},2}^{T_1} \circ \xi_1, \zeta_{\mathrm{Ab},2}^{T_2} \circ \xi_2, \zeta_{\mathrm{Ab},2}^{T_3} \circ \xi_3, \zeta_{\mathrm{Ab},2}^{T_4} \circ \xi_4, \eta)$.*

Proof First set $\beta_i := \zeta_2^{T_i} \circ \xi_i$ and $\alpha_i := \zeta_{\mathrm{Ab},2}^{T_i} \circ \xi_i$ for $i = 1, 2, 3, 4$, so each $\alpha_i \gtrsim \zeta_1^{T_i} \circ \xi_i$ is the maximal Abelian subextension of $\beta_i \gtrsim \zeta_1^{T_i} \circ \xi_i$.

We need to prove that

$$\int_X f_1 f_2 f_3 f_4 g\,\mathrm{d}\mu = \int_X \mathsf{E}_\mu(f_1 \,|\, \alpha_1)\mathsf{E}_\mu(f_2 \,|\, \alpha_2)\mathsf{E}_\mu(f_3 \,|\, \alpha_3)\mathsf{E}_\mu(f_4 \,|\, \alpha_4) g\,\mathrm{d}\mu$$

for any ξ_i-measurable functions f_i and η-measurable function g. In fact it will suffice to prove that

$$\int_X f_1 f_2 f_3 f_4 g\,\mathrm{d}\mu = \int_X f_1 f_2 f_3 \mathsf{E}_\mu(f_4 \,|\, \alpha_4) g\,\mathrm{d}\mu,$$

since then repeating the same argument for the other three isotropy factors in turn completes the proof.

By Lemma 3.20 the three factors $\zeta_1^T \vee \xi_1$, $\zeta_1^T \vee \xi_2$ and $\zeta_1^T \vee \xi_3$ must be joined relatively independently over ζ_1^T. On the other hand, the factor $\xi_4 \vee \eta$ is an extension of ζ_1^T that is certainly still an Abelian isometric extension for the $(\mathbb{Z}\mathbf{n}_4)$-subaction, and so $\xi_1 \vee \xi_2 \vee \xi_3 \vee \zeta_1^T$ must be joined to it relatively independently over

$$\zeta_2^{T^{\mathbf{n}_4}} \wedge \big(\xi_1 \vee \xi_2 \vee \xi_3 \vee \zeta_1^T\big).$$

However, now the Furstenberg-Zimmer Structure Theorem tells us that this last factor must be contained in

$$(\zeta_2^{T^{\mathbf{n}_4}} \wedge \xi_1) \vee (\zeta_2^{T^{\mathbf{n}_4}} \wedge \xi_2) \vee (\zeta_2^{T^{\mathbf{n}_4}} \wedge \xi_3) \vee \zeta_1^T$$

(using that $\zeta_2^{T^{\mathbf{n}_4}} \wedge (\xi_i \vee \zeta_1^T) = (\zeta_2^{T^{\mathbf{n}_4}} \wedge \xi_i) \vee \zeta_1^T$, because ζ_1^T is already one-step distal). Here the factors $\zeta_2^{T^{\mathbf{n}_4}} \wedge \xi_i$ are actually isometric extensions of $\zeta_1^T \wedge \xi_i$ (not just of $\zeta_1^{T^{\mathbf{n}_4}} \wedge \xi_i$), since in each case isometricity for the $(\mathbb{Z}\mathbf{n}_4)$-subaction and *invariance* for the $(\mathbb{Z}\mathbf{n}_i)$-subaction together imply isometricity for the whole $\mathbb{Z}^2$-system $\zeta_1^{T^{\mathbf{n}_4}} \wedge \xi_i$, since $\mathbb{Z}\mathbf{n}_i + \mathbb{Z}\mathbf{n}_4$ has finite index in $\mathbb{Z}^2$ by the non-parallel assumption.

Overall this tells us that $\xi_4 \vee \eta$ is relatively independent from the factors ξ_1, ξ_2 and ξ_3 over their further factors β_1, β_2 and β_3; and now applying the same argument with any of the other isotropy factors as the distinguished factor in place of ξ_4, we deduce that this latter is relatively independent from all our other factors over β_4.

By reducing to the factor of $\mathbf{X}$ generated by the β_i and η, we may therefore assume that each $\mathbf{X}_i$ is itself a two-step distal system (since the join $\beta_1 \vee \beta_2 \vee \beta_3 \vee \beta_4 \vee \eta$ is still two-step distal, and so its maximal isotropy factor in each direction $\mathbf{n}_i$ is also two-step distal and hence equal to β_i).

To make the remaining reduction to have α_i in place of β_i, now let $\mathbf{Z}_1^T = (Z_\star, m_{Z_\star}, \phi_\star)$ be some coordinatization of the Kronecker factor ζ_1^T as a direct integral of ergodic $\mathbb{Z}^2$-group rotations, and let us pick coordinatizations

$$\begin{tikzcd}
\mathbf{X}_i \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "\zeta_1^T|_{\xi_i}"'] & \mathbf{Z}_1^{T_i} \ltimes (G_{i,\bullet}/H_{i,\bullet}, m_{G_{i,\bullet}/H_{i,\bullet}}, \sigma_i) \arrow[d, "\text{canonical}"] \\
& \mathbf{Z}_1^{T_i}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Y} \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\zeta_1^T|_\eta"'] & & \mathbf{Z}_1^T \ltimes (A_\bullet, m_{A_\bullet}, \tau) \arrow[dl, "\text{canonical}"] \\
& \mathbf{Z}_1^T. &
\end{tikzcd}$$

We know this may be done so that the σ_i and τ are relatively ergodic, and so now replacing each $\mathbf{X}_i$ with its covering group extension and joining these relatively independently over the joining $\mathbf{X}$ of the $\mathbf{X}_i$'s and $\mathbf{Y}$, we reduce the problem to the case in which $H_{i,\bullet} = \{1_{G_{i,\bullet}}\}$.

Given this we know that any joining of the above relatively ergodic group extensions of $\mathbf{Z}_1^T$ is described by some $T|_{\zeta_1^T}$-invariant measurable Mackey group data

$$M_z \leq \prod_{i=1}^{4} G_{i,z_i} \times A_z$$

and a section $b : Z \to \prod_{i=1}^4 G_{i,z_i} \times A_z$, where $z \in Z_\star$ and $z_i = \zeta_0^{T^{\mathbf{n}_i}}|_{\zeta_1^T}(z)$. To complete the proof we will show that

$$M_z \geq \prod_{i=1}^{4} [G_{i,z_i}, G_{i,z_i}] \times \{1_{A_z}\}$$

almost surely, since in this case we may quotient out each extension $\mathbf{X}_i \to \mathbf{Z}_1^{T_i}$ fibrewise by the normal subgroups $[G_{i,\bullet}, G_{i,\bullet}] \leq G_{i,\bullet}$ to obtain that our joining is relatively independent over some Abelian subextensions, as required.

The point is that for any three-subset $\{i_1, i_2, i_3\} \subset \{1,2,3,4\}$ the projection of $M_\bullet$ onto the product of factor groups $G_{i_j, z_{i_j}}$, $j = 1,2,3$ is just the Mackey group data of the joining of $\xi_{i_1}, \xi_{i_2}, \xi_{i_3}$ and ζ_1^T as factors of $\mathbf{X}$. By Lemma 3.20 these are relatively independent over ζ_1^T, so this coordinate projection of the Mackey group must be the whole of $\prod_{j=1}^3 G_{i_j, z_{i_j}}$. Hence $M_\bullet$ has full projections onto any three of G_{i,z_i}, and so for any $g_1, h_1 \in G_{1,z_1}$ (say) we can find $g_2 \in G_{2,z_2}$, $h_3 \in G_{3,z_3}$ and $a, b \in A_z$ such that

$$\begin{aligned}
&(g_1, g_2, 1, 1, a), (h_1, 1, h_3, 1, b) \in M_z \\
&\qquad \Rightarrow \qquad [(g_1, g_2, 1, 1, a), (h_1, 1, h_3, 1, b)] = ([g_1, h_1], 1, 1, 1, 1) \in M_z.
\end{aligned}$$

Arguing similarly for the other G_{i,z_i}, we deduce that $M_\bullet$ contains the Cartesian product of commutator subgroups, as required. $\square$

Proposition 3.22. *If h_1, h_2 and h are integers as in Lemma 3.18, $\mathbf{X}$ is an ergodic $\mathbb{Z}^2$-system whose Kronecker factor $\zeta_1^T : \mathbf{X} \to \mathbf{Z}_1^T$ is DIO and $\eta_i : \mathbf{X} \to \mathbf{Y}_i$ is an $((h_i, 1), (h_i, 0), (0, 1))$-directional CL-extension of ζ_1^T for $i = 1, 2$, then the two factors*

$$\zeta_0^{T_1^h} \vee \zeta_0^{T_2} \vee \zeta_0^{T_1^{h_i} T_2} \vee \eta_i \qquad i = 1, 2$$

of $\mathbf{X}$ are relatively independent over a common further factor of the form $\zeta_0^{T_1^h} \vee \zeta_0^{T_2} \vee \eta$ where η has target an $(h\mathbb{Z}^2, (h, 0), (0, h))$-directional CL-system.

Proof Since $\eta_1 \vee \eta_2$ still has target a two-step Abelian isometric system, the preceding lemma shows that $\zeta_0^{T_1^h}$, $\zeta_0^{T_2}$, $\zeta_0^{T_1^{h_1} T_2}$, $\zeta_0^{T_1^{h_2} T_2}$ and $\eta_1 \vee \eta_2$ are all relatively independent over their maximal two-step Abelian factors. Denoting the first four of these by α_1, α_2, $\alpha_{12,1}$ and $\alpha_{12,2}$ respectively, it will therefore suffice to prove that $\alpha_1 \vee \alpha_{12,1} \vee \alpha_2 \vee \eta_1$ and $\alpha_1 \vee \alpha_{12,2} \vee \alpha_2 \vee \eta_2$ are relatively independent over some further common factor $\alpha_1 \vee \alpha_2 \vee \eta$ with η a directional CL-factor of the kind asserted.

However, as described following the introduction of directional CL-systems in Subsection 3.6 of [5], each $\alpha_1 \vee \alpha_{12,i} \vee \alpha_2 \vee \eta_i$ is itself still an $((h_i, 1), (h_i, 0), (0, 1))$-directional CL-system, and so this latter assertion follows at once from Lemma 3.18. This completes the proof. □

We can now make use of the above-found relative independence through the following simple lemma.

Lemma 3.23. *Suppose that (X, μ) is a standard Borel probability space, $\pi_n : X \to Y_n$ is a sequence of factor maps of X and $\alpha_n : Y_n \to Z_n$ is a sequence of further factor maps of Y_n such that (π_n, π_m) are relatively independent over $(\alpha_n \circ \pi_n, \alpha_m \circ \pi_m)$ whenever $n \neq m$ (note that we do not require such relative independence for more than two of the π_i at once). If $f \in L^\infty(\mu)$ is such that $\limsup_{n\to\infty} \|\mathsf{E}_\mu(f \,|\, \pi_n)\|_2 > 0$, then also $\limsup_{n\to\infty} \|\mathsf{E}_\mu(f \,|\, \alpha_n)\|_2 > 0$.*

Proof By thinning out our sequence if necessary, we may assume that for some $\eta > 0$ we have $\|\mathsf{E}_\mu(f \,|\, \pi_n)\|_2 \geq \eta$ for all n. Suppose, for the sake of contradiction, that $\mathsf{E}_\mu(f \,|\, \alpha_n) \to 0$ as $n \to \infty$. Consider the sequence of Hilbert subspaces $L_n \leq L^2(\mu)$ comprising those functions that are π_n-measurable and the further subspaces $K_n \leq L_n$ comprising those that are α_n-measurable. Then by assumption all the subspaces $L_n \ominus K_n$ are mutually orthogonal, but f has orthogonal projection of norm at least $\eta/2$ onto all but finitely many of them, which is clearly impossible. □

Proof of Theorem 1.3 Letting $\pi : \mathbf{X} \to \mathbf{X}_0$ be the ergodic pleasant extension for triple linear averages in general position obtained by applying Theorem 1.2 and Lemma 3.12 and then making a further extension of the Kronecker factor using Lemma 3.16 if necessary, now Corollary 3.14, Proposition 3.22 and Lemma 3.23 show that whenever $f_1, f_2 \in L^\infty(\mu)$ have $S_n(f_1, f_2) \not\to 0$, they also satisfy $\mathsf{E}_\mu(f_i \,|\, \zeta_{\mathrm{pro}}^{T_1} \vee \zeta_0^{T_2} \vee \eta_\infty) \neq 0$ where $\zeta_{\mathrm{pro}}^{T_1}$ is the factor generated by all $\zeta_0^{T_1^h}$, $h \geq 1$, and η_∞ is a join over some sequence of integers h of $(h\mathbb{Z}^2, (h,0), (0,h))$-directional CL-factors. Writing $\xi := \zeta_{\mathrm{pro}}^{T_1} \vee \zeta_0^{T_2} \vee \eta_\infty$, the proposition follows at once by considering the decomposition

$$\begin{aligned} S_N(f_1, f_2) &= S_N(\mathsf{E}_\mu(f_1 \,|\, \xi), \mathsf{E}_\mu(f_2 \,|\, \xi)) \\ &\quad + S_N(f_1 - \mathsf{E}_\mu(f_1 \,|\, \xi), \mathsf{E}_\mu(f_2 \,|\, \xi)) + S_N(f_1, f_2 - \mathsf{E}_\mu(f_2 \,|\, \xi)). \end{aligned}$$

□

3.3 Second reduction

Theorem 1.3 shows that Theorem 1.1 will follow if we prove that $S_N(f_1, f_2)$ converges whenever f_i is ξ_i-measurable. By approximation in $L^2(\mu)$ and multilinearity, it actually suffices to consider the averages $S_N(f_{11}f_{12}g_1, f_{21}f_{22}g_2)$ in which each f_{j1} is $\zeta_0^{T_1^\ell}$-measurable for some large $\ell \geq 1$, each f_{j2} is $\zeta_0^{T_2}$-measurable and each g_j is η-measurable for some $(h\mathbb{Z}^2, (h,0), (0,h))$-directional CL-factor η for some large $h \geq 1$.

Next, writing

$$\begin{aligned} S_N(f_{11}f_{12}g_1, f_{21}f_{22}g_2) &= \frac{1}{N}\sum_{n=1}^{N}((f_{11}\cdot f_{12}\cdot g_1)\circ T_1^{n^2})((f_{21}\cdot f_{22}\cdot g_2)\circ T_1^{n^2}T_2^{n}) \\ &\sim \frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{1}{(N/\ell)}\sum_{n=1}^{\lfloor N/\ell\rfloor}((f_{11}\cdot f_{12}\cdot g_1)\circ T_1^{(\ell n+k)^2})((f_{21}\cdot f_{22}\cdot g_2)\circ T_1^{(\ell n+k)^2}T_2^{\ell n+k}) \\ &= \frac{1}{\ell}\sum_{k=0}^{\ell-1}(f_{11}\circ T_1^{k^2})\Big(\frac{1}{(N/\ell)}\sum_{n=1}^{\lfloor N/\ell\rfloor}((f_{12}\cdot g_1)\circ T_1^{(\ell n+k)^2})((f_{21}\cdot f_{22}\cdot g_2)\circ T_1^{(\ell n+k)^2}T_2^{\ell n+k})\Big) \\ &= \frac{1}{\ell}\sum_{k=0}^{\ell-1}(f_{11}\circ T_1^{k^2})\Big(\frac{1}{(N/\ell)}\sum_{n=1}^{\lfloor N/\ell\rfloor}((f_{12}\cdot f_{22}\cdot g_1)\circ T_1^{(\ell n+k)^2}) \\ &\qquad\qquad \cdot(g_2\circ T_1^{(\ell n+k)^2}T_2^{\ell n+k})(f_{21}\circ T_1^{k^2}\circ T_2^{\ell n+k})\Big) \end{aligned}$$

(recalling that $\sim$ denotes asymptotic agreement in $L^2(\mu)$ as $N \to \infty$), we see that it will suffice to prove convergence in $L^2(\mu)$ for all averages along infinite arithmetic progressions of the form

$$\frac{1}{(N/\ell)}\sum_{n=1}^{\lfloor N/\ell \rfloor}((f_{12}\cdot f_{22}\cdot g_1)\circ T_1^{(\ell n)^2+2k(\ell n)})(g_2\circ T_1^{(\ell n)^2+2k(\ell n)}T_2^{\ell n})(f_{21}\circ T_2^{\ell n})$$

for all $k \in \{0,1,\ldots,\ell-1\}$, where for a fixed k we have re-written $(f_{12}\cdot f_{22}\cdot g_1)\circ T_1^{k^2}$ as simply $f_{12}\cdot f_{22}\cdot g_1$ and similarly for the other factors, and have discarded the initial multiplication by the n-independent function $f_{11}\circ T_1^{k^2}T_2^k$.

If we now simply re-label T_i^ℓ as T_i (and so effectively restrict our attention to the subaction of $\ell\mathbb{Z}^2$), then the above averages are modified to

$$\frac{1}{(N/\ell)}\sum_{n=1}^{\lfloor N/\ell \rfloor}((f_{12}\cdot f_{22}\cdot g_1)\circ T_1^{\ell n^2+2kn})(g_2\circ T_1^{\ell n^2+2kn}T_2^{n})(f_{21}\circ T_2^{n})$$

and now f_{21} is simply T_1-invariant. Moreover, it is clear that any $(h\mathbb{Z}^2,(h,0),(0,h))$-directional CL-system for the action T retains this property under this re-labeling (indeed, the same property for the re-labeled system is potentially slightly weaker), and also if we then restrict attention to any one of the (finitely many) $\ell\mathbb{Z}^2$-ergodic components of the overall system.

Thus, we have now reduced our task to the proof of convergence for averages of the form

$$\frac{1}{N}\sum_{n=1}^{N}((F_2\cdot g_1)\circ T_1^{\ell n^2+an})(g_2\circ T_1^{\ell n^2+an}T_2^{n})(F_1\circ T_2^{n}),$$

for any fixed integers $\ell, a \geq 1$, where F_2 is T_2-invariant, F_1 is T_1-invariant and g_1, g_2 are η-measurable.

This conclusion was obtained by simply re-writing the expression for S_N for the functions of interest to us (with a little sleight of hand to deal with the rational spectrum of T_1). However, it turns out that we can do better still with just a little more work: to wit, that we may also remove the function F_2 from consideration, and so reduce Theorem 1.1 to Proposition 3.25 below. This will rely on the following results from [4, 5].

Proposition 3.24 (The Furstenberg self-joining controls nonconventional averages)**.** *If $f_1, f_2, f_3 \in L^\infty(\mu)$ and*

$$\frac{1}{N}\sum_{n=1}^{N}(f_1\circ T_1^{2\ell hn})(f_2\circ T_1^{2\ell hn}T_2^{-n})(f_3\circ T_2^{-n}) \not\to 0$$

as $N \to \infty$, then there is some $(T_1^{2\ell h} \times T_1^{2\ell h}T_2^{-1} \times T_2^{-1})$-invariant bounded Borel function $G : X^3 \to \mathbb{R}$ such that

$$\int_{X^3} (f_1 \otimes f_2 \otimes f_3) \cdot G \,\mathrm{d}\mu_h^{\mathrm{F}} \neq 0,$$

where $\mu_h^{\mathrm{F}} := \mu^{\mathrm{F}}_{T_1^{2\ell h}, T_1^{2\ell h}T_2^{-1}, T_2^{-1}}$ is the Furstenberg self-joining (see Subsection 4.1 of [4]). This is a three-fold self-joining of $(X, \mu, T_1^{2\ell h}, T_2)$ that is also invariant under the transformation $\vec{T}_h := T_1^{2\ell h} \times T_1^{2\ell h}T_2^{-1} \times T_2^{-1}$, and has the following properties:

- *The restriction of μ_h^{F} to Z^3 is the Haar measure m_{Z_h} of some closed subgroup $Z_h \leq Z^3$, and if the Kronecker factor (Z, m_Z, ϕ) of $\mathbf{X}$ is DIO then*

$$\begin{aligned} Z_h &= \{(z_1, z_2, z_3) \in Z^3 :\ z_1 z_2^{-1} \in K_{(0,1)},\ z_1 z_3^{-1} \in K_{(2\ell h,1)},\ z_2 z_3^{-1} \in K_{(2\ell h,0)}\} \\ &= \{(zu, zuv, zv) :\ z \in Z,\ u \in K_{(2\ell h,0)},\ v \in K_{(0,1)},\ uv^{-1} \in K_{(2\ell h,1)}\}, \end{aligned}$$

  where as usual we write $K_{\mathbf{n}} := \overline{\phi(\mathbb{Z}\mathbf{n})}$.

- *The $\vec{T}_h$-ergodic components of the restriction of μ_h^{F} to $(Z \ltimes A)^3$ are almost all of the form*

$$m_{z_0 \cdot \overline{(\phi(2\ell h \mathbf{e}_1), \phi(2\ell h \mathbf{e}_1 - \mathbf{e}_2), \phi(-\mathbf{e}_2))^{\mathbb{Z}}}} \ltimes m_{b_h(\bullet)^{-1} \cdot M_h \cdot a}$$

  for some Mackey group $M_h \leq A^3$ on Z_h, some Borel section $b_h : Z_h \to A^3$ and some fixed $a \in A^3$ and $z_0 \in Z_h$.

These last conclusions follow from the conjunction of Propositions 4.6 and 4.7 in [5] and the discussion of Subsection 4.8 of [5], except for the fact that the Mackey group M_h is constant which results from the presence of the restrictions of the transformations $(T^{\mathbf{n}})^{\times 3}$ to $(Z \ltimes A)^3$ that are described by A-valued cocycles and leave M_h invariant, as in the proof of Proposition 4.10 of [5]. □

Proposition 3.25. *If $\mathbf{X}$ is a $\mathbb{Z}^2$-system as output by Theorem 1.3 and $\ell, a \geq 1$ are fixed integers then the nonconventional ergodic averages*

$$S'_N(g_1, g_2, f) := \frac{1}{N} \sum_{n=1}^{N} (g_1 \circ T_1^{\ell n^2 + an})(g_2 \circ T_1^{\ell n^2 + an} T_2^n)(f \circ T_2^n)$$

converge in $L^2(\mu)$ as $N \to \infty$ whenever g_1, g_2 are η-measurable and f is T_1-invariant.

Proof of Theorem 1.1 from Proposition 3.25 Theorem 1.3 and the re-arrangement above show that it suffices to prove convergence for averages of the form

$$\frac{1}{N}\sum_{n=1}^{N}((F_2\cdot g_1)\circ T_1^{\ell n^2+an})(g_2\circ T_1^{\ell n^2+an}T_2^n)(F_1\circ T_2^n)$$

with F_2 being T_2-invariant and F_1 being T_1-invariant. We will now show that these tend to 0 in $L^2(\mu)$ if F_2 is orthogonal to $\zeta^T_{\mathrm{Ab},2}$, which combined with the T_2-invariance of F_2 shows that it suffices to treat the case when F_2 is actually measurable with respect to $\zeta^T_{\mathrm{Ab},2}\wedge\zeta_0^{T_2}$, which is another $(h\mathbb{Z}^2,(h,0),(0,h))$-directional CL-system and so may be subsumed into the factor η. The resulting averages will then be easily re-arranged into the form S'_N.

By another appeal to the van der Corput estimate we know that the above averages tend to zero in $L^2(\mu)$ unless also

$$\begin{aligned}\frac{1}{H}\frac{1}{N}\sum_{h=1}^{H}\sum_{n=1}^{N}\int_X &(F_2\circ T_1^{\ell n^2+2\ell hn+\ell h^2+an+ah})(\overline{F_2}\circ T_1^{\ell n^2+an})(F_1\circ T_2^{n+h})(\overline{F_1}\circ T_2^n)\\ &\cdot(g_1\circ T_1^{\ell n^2+2\ell hn+\ell h^2+an+ah})(\overline{g_1}\circ T_1^{\ell n^2+an})\\ &\cdot(g_2\circ T_1^{\ell n^2+2\ell hn+\ell h^2+an+ah}T_2^{n+h})(\overline{g_2}\circ T_1^{\ell n^2+an}T_2^n)\,\mathrm{d}\mu\not\to 0\end{aligned}$$

as $N\to\infty$ and then $H\to\infty$.

Using the invariances of the F_i we can change variables in each of the integrals appearing above by $T_1^{-\ell n^2-an}T_2^{-n}$ and find that the above conclusion simplifies to

$$\begin{aligned}&\frac{1}{H}\frac{1}{N}\sum_{h=1}^{H}\sum_{n=1}^{N}\int_X (F_2\circ T_1^{2\ell hn+\ell h^2+ah})\cdot\overline{F_2}\cdot(F_1\circ T_2^h)\cdot\overline{F_1}\\ &\qquad\cdot(g_1\circ T_1^{2\ell hn+\ell h^2+ah}T_2^{-n})(\overline{g_1}\circ T_2^{-n})(g_2\circ T_1^{2\ell hn+\ell h^2+ah}T_2^h)\overline{g_2}\,\mathrm{d}\mu\\ &=\frac{1}{H}\frac{1}{N}\sum_{h=1}^{H}\sum_{n=1}^{N}\int_X((F_2\cdot(g_2\circ T_2^h))\circ T_1^{2\ell hn+\ell h^2+ah})(g_1\circ T_1^{2\ell hn+\ell h^2+ah}T_2^{-n})(\overline{g_1}\circ T_2^{-n})\\ &\qquad\qquad\cdot\overline{F_2}\cdot(F_1\circ T_2^h)\cdot\overline{F_1}\cdot\overline{g_2}\,\mathrm{d}\mu\not\to 0.\end{aligned}$$

Hence, extracting the active part of the average over $n\in\{1,2,\ldots,N\}$ it follows that for some $h\geq 1$ (here we need only one such value) we have

$$\frac{1}{N}\sum_{n=1}^{N}((F_2\cdot(g_2\circ T_2^h))\circ T_1^{2\ell hn+\ell h^2+ah})(g_1\circ T_1^{2\ell hn+\ell h^2+ah}T_2^{-n})(\overline{g_1}\circ T_2^{-n})\not\to 0$$

in $L^2(\mu)$.

This is another instance of the kind of triple linear average that we have considered previously, but now with functions $F_2\cdot(g_2\circ T_2^h)$, $g_1\circ T_1^{\ell h^2+ah}$ and $\overline{g_1}$ that are measurable with respect to more restricted factors of the overall system $\mathbf{X}$. Applying Proposition 3.24 we obtain

$$\int_{X^3}\big(((F_2\cdot(g_2\circ T_2^h))\circ T_1^{\ell h^2+ah})\otimes(g_1\circ T_1^{\ell h^2+ah})\otimes\overline{g_1}\big)\cdot G\,\mathrm{d}\mu^{\mathrm{F}}\neq 0$$

for some function $G\in L^\infty(\mu^{\mathrm{F}})$ that is invariant under $\vec{T}:=T_1^{2\ell h}\times T_1^{2\ell h}T_2^{-1}\times T_2^{-1}$.

Let π_1, π_2 and π_3 be the three coordinate projections $X^3\to X$, and now consider on (X^3,μ^{F}) the two μ^{F}-preserving transformations $\vec{T}$ and $T_2^{\times 3}$. The function $(F_2\circ T_1^{\ell h^2+ah})\circ\pi_1$ is $T_2^{\times 3}$-invariant (simply because F_2 was assumed T_2-invariant), and the above nonvanishing integral asserts that this function has a positive inner product with the function

$$(g_2\circ T_2^h\circ T_1^{\ell h^2+ah}\circ\pi_1)\cdot(((g_1\circ T_1^{\ell h^2+ah})\cdot\overline{g_1})\circ\pi_2)\cdot G,$$

where $g_2\circ T_2^h\circ T_1^{\ell h^2+ah}\circ\pi_1$ and $((g_1\circ T_1^{\ell h^2+ah})\cdot\overline{g_1})\circ\pi_2$ are both measurable with respect to some two-step Abelian factor by assumption and where G is $\vec{T}$-invariant. Moreover $\vec{T}$ simply restricts to $T_1^{2\ell h}$ under π_1. Therefore Lemma 3.21 above implies that the factor $\zeta_0^{T_2}\circ\pi_1\lesssim\zeta_0^{T_2^{\times 3}}$ of $\mathbf{X}':=(X^3,\mu^{\mathrm{F}},\vec{T},T_2^{\times 3})$ is relatively independent from $\zeta_{\mathrm{Ab},2}^{T_2^{\times 3},\vec{T}}\vee\zeta_0^{\vec{T}}$ over the two-step Abelian factor $\zeta_{\mathrm{Ab},2}^{\vec{T},T_2^{\times 3}}\wedge\zeta_0^{T_2^{\times 3}}$. This, in turn, is a two-step Abelian isometric system on which $T_2^{\times 3}$ is invariant, and so it must be joined to $\zeta_0^{T_2}\circ\pi_1$ relatively independently over the maximal two-step Abelian factor of $\zeta_0^{T_2}\circ\pi_1$. It follows that F_2 must have nonzero conditional expectation onto the factor $\zeta_{\mathrm{Ab},2}^{T}\wedge\zeta_0^{T_2}$, as claimed.

Since this last factor is also a $(h\mathbb{Z}^2,(h,0),(0,h))$-directional CL-system, we may assume that it is already contained in η, and therefore we have shown that it suffices to prove convergence of our averages when we write simply g_1 in place of $F_2\cdot g_1$. These puts them into the form $S_N'(g_1,g_2,F_1)$ treated by Proposition 3.25, and so completes the proof. $\square$

By continuing in the vein of the above proof we could try to obtain also a simplification of the function F_1. However, in fact these methods do not seem to give a reduction of this function that is strong enough to be useful. In the next subsections we will change tack to give a different kind of simplification of the averages, from which convergence can be proved given no further information about the function F_1.

3.4 Using the Mackey group of the Furstenberg self-joining

The last subsection has left us to consider the averages

$$S'_N(g_1,g_2,f) := \frac{1}{N}\sum_{n=1}^N (g_1\circ T_1^{\ell n^2+an})(g_2\circ T_1^{\ell n^2+an}T_2^n)(f\circ T_2^n)$$

for g_1,g_2 that are measurable with respect to some $(m\mathbb{Z}^2,(m,0),(0,m))$-directional CL-factor $\eta:\mathbf{X}\to\mathbf{Y}$ and f that is T_1-invariant. Let us pick a coordinatization of the directional CL-factor, say as $\eta:\mathbf{X}\to(Z,m_Z,\phi)\ltimes(A,m_A,\sigma)$ for some compact metrizable Abelian groups Z and A, a dense homomorphism $\phi:\mathbb{Z}^2\to Z$ and a cocycle $\sigma:\mathbb{Z}^2\times Z\to A$ over R_ϕ, chosen so that the canonical further factor onto (Z,m_Z,ϕ) is the whole Kronecker factor. By Lemma 3.16 we may assume that (Z,m_Z,ϕ) has the DIO property.

In these terms, again by L^2-continuity and multilinearity, to prove convergence of these averages it suffices to consider functions $g_i(z,a)$ of the form $\kappa_i(z)\chi_i(a)$ with $\kappa_i\in\widehat{Z}$ and $\chi_i\in\widehat{A}$ for $i=1,2$. We will refer to functions of this form as **vertical eigenfunctions** of the system $(Z,m_Z,\phi)\ltimes(A,m_A,\sigma)$, and will refer to the characters χ_i appearing in their definition as their associated **vertical characters**. For these functions our averages become

$$\begin{aligned}
&S'_N(g_1,g_2,f)(x)\\
&=\frac{1}{N}\sum_{n=1}^N\kappa_1(\phi(\ell n^2+an,0)z)\cdot\chi_1(a)\cdot\chi_1(\sigma((\ell n^2+an,0),z))\\
&\qquad\qquad\cdot\kappa_2(\phi(\ell n^2+an,n)z)\cdot\chi_2(a)\cdot\chi_2(\sigma((\ell n^2+an,n),z))\cdot f(T_2^n(x))\\
&=\kappa_1(z)\chi_1(a)\kappa_2(z)\chi_2(a)\frac{1}{N}\sum_{n=1}^N\kappa_1(\phi(\mathbf{e}_1))^{\ell n^2+an}\kappa_2(\phi(\mathbf{e}_1))^{\ell n^2+an}\kappa_2(\phi(\mathbf{e}_2))^n\\
&\qquad\qquad\cdot\chi_1(\sigma((\ell n^2+an,0),z))\cdot\chi_2(\sigma((\ell n^2+an,n),z))\cdot f(T_2^n(x))
\end{aligned}$$

where we write $(z,a):=\eta(x)$ and have used that κ_i and χ_i are characters. Writing $\theta_1:=(\kappa_1\cdot\kappa_2)(\phi(\mathbf{e}_1))$ and $\theta_2:=\kappa_2(\phi(\mathbf{e}_2))$, we immediately deduce the following.

Lemma 3.26. *The averages $S'_N(g_1,g_2,f)$ of Proposition 3.25 all converge in $L^2(\mu)$ as $N\to\infty$ if and only if this is true of the averages*

$$\frac{1}{N}\sum_{n=1}^N\theta_1^{\ell n^2+an}\theta_2^n\cdot\chi_1(\sigma((\ell n^2+an,0),z))\cdot\chi_2(\sigma((\ell n^2+an,n),z))\cdot f(T_2^n(x))$$

for any $\theta_1,\theta_2\in\mathrm{S}^1$. □

In the conclusion of this lemma it is clear that the remaining 'awkwardness' for the purposes of proving norm convergence resides in the expression

$$\chi_1(\sigma((\ell n^2+an,0),z))\cdot\chi_2(\sigma((\ell n^2+an,n),z)).$$

This is a sequence of functions on the group rotation factor Z whose behaviour as n varies we have yet to control with much precision. Most of the remainder of the proof will be directed towards exerting such control. In our approach to this we will follow the basic strategy used by Host and Kra in [16] of arguing that if our averages do not tend to 0 in $L^2(\mu)$, then the cocycle σ must give rise to some nontrivial Mackey data, and hence a nontrivial combined cocycle equation, inside the Furstenberg self-joining; and then using that equation itself to analyze the behaviour of expressions such as our product of cocycles above. However, the details of our implementation of this approach are rather different from Host and Kra's, and in particular will rest on much of our earlier study of directional CL-systems.

To begin the next stage of our analysis, we once again apply the van der Corput estimate. Letting $u_n := (g_1\circ T_1^{\ell n^2+an})(g_2\circ T_1^{\ell n^2+an}T_2^n)(f\circ T_2^n)$, we deduce as before that either $S'_N(g_1,g_2,f)\to 0$ in $L^2(\mu)$ or else we also have

$$\begin{aligned}\frac{1}{H}\sum_{h=1}^{H}\frac{1}{N}\sum_{n=1}^{N}\int_X &(g_1\circ T_1^{\ell n^2+2\ell nh+\ell h^2+an+ah})(\overline{g_1}\circ T_1^{\ell n^2+an})\\ &\cdot(g_2\circ T_1^{\ell n^2+2\ell nh+\ell h^2+an+ah}T_2^{n+h})(\overline{g_2}\circ T_1^{\ell n^2+an}T_2^{n})\\ &\cdot(f\circ T_2^{n+h})(\overline{f}\circ T_2^{n})\,\mathrm{d}\mu\not\to 0\end{aligned}$$

as $N\to\infty$ and then $H\to\infty$; and now, still as in the previous section, using the T_1-invariance of f we can change variables in these integrals by $T_1^{-\ell n^2-an}T_2^{-n}$ (and change the order of some of the factors) to obtain

$$\begin{aligned}\frac{1}{H}\sum_{h=1}^{H}\frac{1}{N}\sum_{n=1}^{N}\int_X &(g_2\circ T_1^{2\ell nh+\ell h^2+ah}T_2^{h})(g_1\circ T_1^{2\ell nh+\ell h^2+ah}T_2^{-n})(\overline{g_1}\circ T_2^{-n})\\ &\cdot\overline{g_2}\cdot(f\circ T_2^{h})\cdot\overline{f}\,\mathrm{d}\mu\not\to 0,\end{aligned}$$

and this implies that for some $\varepsilon>0$ we have

$$\Big\|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(g_2\circ T_1^{2\ell nh+\ell h^2+ah}T_2^{h})(g_1\circ T_1^{2\ell nh+\ell h^2+ah}T_2^{-n})(\overline{g_1}\circ T_2^{-n})\Big\|_2^2\geq\varepsilon$$

for infinitely many integers $h\geq 1$.

At this point another appeal to Proposition 3.24 implies that for infinitely many integers $h \geq 1$ the function $(g_2 \circ T_1^{\ell h^2+ah}T_2^h) \otimes (g_1 \circ T_1^{\ell h^2+ah}) \otimes \overline{g_1}$ has nonzero conditional expectation onto the $\vec{T}_h$-invariant factor of $(X^3, \mu_h^{\mathrm{F}})$, where $\vec{T}_h := T_1^{2\ell h} \times (T_1^{2\ell h}T_2^{-1}) \times T_2^{-1}$ as in that proposition. This is essentially the same conclusion that was used for our first reduction above, except that our change-of-variables above was slightly different this time (there we changed by $T_1^{-\ell n^2-an}$, rather than $T_1^{-\ell n^2-an}T_2^{-n}$), and this has led here to a different triple of directions.

Nevertheless, they are still in general position with the origin, and so we can make use of the description of the restriction of μ_h^{F} to $(Z \ltimes A)^3$ given in Proposition 3.24. Observe also that

$$g_2 \circ T_1^{\ell h^2+ah}T_2^h(z,a) = \kappa_2(\phi(\ell h^2+ah,h)z) \cdot \chi_2(\sigma((\ell h^2+ah,h),z)) \cdot \chi_2(a)$$

is still a vertical eigenfunction with vertical character χ_2, and similarly $g_1 \circ T_1^{\ell h^2+ah}$ and $\overline{g_1}$. Combining this with the description of the $\vec{T}_h$-ergodic components of μ_h^{F} given in Proposition 3.24, it follows that if $(g_2 \circ T_1^{\ell h^2+ah}T_2^h) \otimes (g_1 \circ T_1^{\ell h^2+ah}) \otimes \overline{g_1}$ has nontrivial conditional expectation onto the $\vec{T}_h$-invariant factor then the character $\chi_2 \otimes \chi_1 \otimes \overline{\chi_1}$ must have nonzero average over the Mackey group $M_h \leq A^3$. Combining this with our other conclusions leads to the following.

Lemma 3.27. *For any h for which the above averages do not tend to zero we must have*

$$M_h \leq \ker(\chi_2 \otimes \chi_1 \otimes \overline{\chi_1})$$

where M_h is the Mackey group given by Proposition 3.24, and so its Mackey section quotients to give a Borel function $b_h : Z_h \to \mathrm{S}^1$ such that

$$\begin{aligned}\chi_2 \circ \sigma((2\ell h,0),z_1) \cdot \chi_1 \circ \sigma((2\ell h,-1),z_2) \cdot \overline{\chi_1} \circ \sigma((0,-1),z_3)\\ = \Delta_{(\phi(2\ell h\mathbf{e}_1),\phi(2\ell h\mathbf{e}_1-\mathbf{e}_2),\phi(-\mathbf{e}_2))}b_h(z_1,z_2,z_3)\end{aligned}$$

for Haar-a.e. $(z_1,z_2,z_3) \in Z_h$. □

We will soon argue that given any two different values of h, say h_1 and h_2, for which the conclusion of Lemma 3.27 holds, we can use the structure of directional CL-systems in conjunction with the above combined coboundary equations to give some useful information for our combined cocycle on a subgroup of Z^3 that is 'effectively' much larger than either of Z_{h_1} or Z_{h_2} individually, and for a whole finite-index subgroup $\Gamma \leq \mathbb{Z}^2$.

3.5 Using several combined coboundary equations

The following is another useful consequence of the DIO property.

Lemma 3.28. *If (Z, m_Z, ϕ) has the DIO property and $\mathbf{n}_1$, $\mathbf{n}_2 \in \mathbb{Z}^2$ are linearly independent then there is a unique continuous isomorphism $\gamma_{\mathbf{n}_1,\mathbf{n}_2} : K_{\mathbf{n}_1} \to K_{\mathbf{n}_2}$ such that the map*

$$u \mapsto u \cdot \gamma_{\mathbf{n}_1,\mathbf{n}_2}(u)$$

is an isomorphism $K_{\mathbf{n}_1} \to K_{\mathbf{n}_1+\mathbf{n}_2}$.

Proof Since $\mathbf{n}_1 = (\mathbf{n}_1 + \mathbf{n}_2) - \mathbf{n}_2$ it follows that $K_{\mathbf{n}_1} \leq K_{\mathbf{n}_1+\mathbf{n}_2} \cdot K_{\mathbf{n}_2}$. Hence for any $u \in K_{\mathbf{n}_1}$ there are $w \in K_{\mathbf{n}_1+\mathbf{n}_2}$ and $v \in K_{\mathbf{n}_2}$ such that $u = wv^{-1}$, and moreover the DIO property implies that $K_{\mathbf{n}_1+\mathbf{n}_2} \cap K_{\mathbf{n}_2} = \{1_Z\}$ and so these w and v are uniquely determined. Now setting $\gamma_{\mathbf{n}_1,\mathbf{n}_2}(u) := v$ it follows easily from uniqueness that this is a continuous homomorphism, and that it has the analogously-defined map $\gamma_{\mathbf{n}_2,\mathbf{n}_1}$ for an inverse and so is an isomorphism. Finally, we can check similarly that the map

$$u \mapsto u \cdot \gamma_{\mathbf{n}_1,\mathbf{n}_2}(u)$$

simply gives the analogously-defined map $\gamma_{\mathbf{n}_1,\mathbf{n}_1+\mathbf{n}_2}$ so it is also a continuous isomorphism. This completes the proof. $\square$

We now introduce the 'essentially larger' subgroup of Z^3 where we will still be able to establish some useful structure to our combined cocycle. Recalling that the target of η is an $(m\mathbb{Z}^2, m\mathbf{e}_1, m\mathbf{e}_2)$-directional CL-system for some $m \geq 1$, and given two distinct integers h_1 and h_2 satisfying the conclusions of Lemma 3.27, let $h := 2\ell \cdot \mathrm{l.c.m.}(m, h_1, h_2, h_1 + h_2, h_1 - h_2)$, and let

$$\tilde{Z}_0 := \{(zu, zuv, zv) : \ z \in Z,\ u \in K_{h\mathbf{e}_1},\ v \in K_{h\mathbf{e}_2}\}.$$

It is easy to see that $\tilde{Z}_0 \cap Z_{h_i}$ is always of finite index in Z_{h_i} for $i = 1, 2$: indeed, if $(zu, zuv, zv) \in Z_{h_i}$ then there is always some $k \in \{0, 1, \ldots, h\}$ for which $(zu, zuv, zv) \cdot (\phi(2\ell h_i\mathbf{e}_1), \phi(2\ell h_i\mathbf{e}_1 + \mathbf{e}_2), \phi(\mathbf{e}_2))^k \in \tilde{Z}_0$. On the other hand, this intersection can be of infinite index in $\tilde{Z}_0$.

Let $\psi : \mathbb{Z}^2 \to Z^3$ be the homomorphism $(n_1, n_2) \mapsto (\phi(n_1\mathbf{e}_1), \phi(n_1\mathbf{e}_1+n_2\mathbf{e}_2), \phi(n_2\mathbf{e}_2))$. Also, by restricting from our $\mathbb{Z}^2$-action to any of the (finitely many) ergodic components of the subaction of $h\mathbb{Z}^2$, and observing that all of the structural information we have accrued so far is preserved, we may assume that the subaction of $h\mathbb{Z}^2$ is ergodic.

We will show that given the two combined coboundary equations from Lemma 3.27 for h_1 and h_2 and also the previously-obtained structure of a directional CL-system, we can actually obtain some useful information on the combined cocycle over R_ψ for the whole of the further finite-index subgroup $\Gamma := \mathbb{Z}(2\ell h_1 h, h)+\mathbb{Z}(2\ell h_2 h, h) \leq h\mathbb{Z}^2$.

Lemma 3.29. *For any integers h_1, h_2 and h satisfying the conclusion of Lemma 3.27 there are a Borel maps $\tilde{b}_i : \tilde{Z}_0 \to \mathrm{S}^1$ and $\tilde{c}_i : K_{h\mathbf{e}_1} \times K_{h\mathbf{e}_2} \to \mathrm{S}^1$ for $i = 1,2$ such that $\tilde{c}_i$ takes the special form of the functions output by Proposition 2.1, and*

$$\begin{aligned}\chi_2 \circ \sigma(2\ell h_i h\mathbf{e}_1, zu) \cdot \chi_1 \circ \sigma(2\ell h_i h\mathbf{e}_1 - h\mathbf{e}_2, zuv) \cdot \overline{\chi_1} \circ \sigma(-h\mathbf{e}_2, zu)& \\ = \Delta_{\psi(2\ell h_i h,h)}\tilde{b}_i(zu, zuv, zv) \cdot \tilde{c}_i(u,v)&\end{aligned}$$

for Haar-a.e. $(zu, zuv, zv) \in \tilde{Z}_0$.

Proof First note that

$$R_{\psi(n_1,n_2)}(zu, zuv, zv) = \big(z(u\phi(n_1\mathbf{e}_2)), z(u\phi(n_1\mathbf{e}_1))(v\phi(n_2\mathbf{e}_2)), z(v\phi(n_2\mathbf{e}_2))\big).$$

As a result, the above combined cocycle equation can be regarded separately for each fixed value of z as an equation involving only the variables u and v. Therefore it suffices to prove instead the existence of maps $\tilde{c}_i$ satisfying the above equations that are simply Borel, $R_{(\phi(2\ell h_i h,0),\phi(0,-h))}$-invariant and do not depend on z, since we can then choose some generic $z \in Z$ and apply Proposition 2.1 to the resulting combined cocycle equations for that fixed z to modify each $\tilde{c}_i$ into the desired special form.

Having observed this, the proof that there are Borel maps $\tilde{b}_i$ and $\tilde{c}_i$ of this form satisfying the above equation will not involve the fact that we are assuming ourselves given two distinct values of h_i as output by be Lemma 3.27; the only appeal we make to this fact is in this initial application of Proposition 2.1.

Let us write

$$\begin{aligned}&\tau_i(zu, zuv, zv)\\ &\quad := \chi_2 \circ \sigma(2\ell h_i h\mathbf{e}_1, zu) \cdot \chi_1 \circ \sigma(2\ell h_i h\mathbf{e}_1 - h\mathbf{e}_2, zuv) \cdot \overline{\chi_1} \circ \sigma(-h\mathbf{e}_2, zv).\end{aligned}$$

We will need the isomorphisms given by Lemma 3.28. In particular, let $\gamma_i : K_{(0,h)} \to K_{(2\ell h h_i,0)}$ be such that $v\gamma_i(v)^{-1} \in K_{(2\ell h h_i,-h)}$ for all $v \in K_{(0,h)}$.

For any $(zu, zuv, zv) \in \tilde{Z}_0$ consider the decomposition

$$\begin{aligned}
&\tau_i(zu, zuv, zv) \\
&\qquad = \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v) \cdot \overline{\left(\overline{\tau_i(zu, zuv, zv)} \cdot \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v)\right)}.
\end{aligned}$$

We will examine the two factors on the right-hand side of this decomposition separately.

On the one hand, by the construction of γ_i we know that $(zu, zuv, zu\gamma_i(v)^{-1}v) \in Z_{h_i}$ and that the map $\tilde{Z}_0 \to Z_{h_i} : (zu, zuv, zv) \mapsto (zu, zuv, zu\gamma_i(v)^{-1}v)$ is a homomorphism that covers a finite-index (and so positive-measure) subgroup of Z_{h_i}, because by the uniqueness of γ_i it must be the identity on $\tilde{Z}_0 \cap Z_{h_i}$. Hence by Lemma 3.27 we have

$$\tau_i(zu, zuv, zu\gamma_i(v)^{-1}v) = (\Delta_{\psi(2\ell h_i h, -h)} b_{h_i})(zu, zuv, zu\gamma_i(v)^{-1}v)$$

for $m_{\tilde{Z}_0}$-a.e. (zu, zuv, zv). Since we must have $\gamma_i(\phi(0, -h)) = \phi(2\ell h_i h, 0)$, again by the uniqueness of γ_i, and therefore $\phi(2\ell h_i h, 0)\gamma_i(\phi(0, -h))^{-1} = 1$, if we define

$$b'_i(zu, zuv, zv) := b_{h_i}(zu, zuv, zu\gamma_i(v)^{-1}v)$$

then it follows that

$$\begin{aligned}
&\Delta_{\psi(2\ell h_i h, -h)} b'_i(zu, zuv, zv) \\
&= b'_i(zu \cdot \phi(2\ell h_i h, 0), zuv \cdot \phi(2\ell h_i h, -h), zv \cdot \phi(0, -h)) \cdot \overline{b'_i(zu, zuv, zv)} \\
&= b_{h_i}(zu \cdot \phi(2\ell h_i h, 0), zuv \cdot \phi(2\ell h_i h, -h), zu\gamma_i(v)^{-1}v \cdot \phi(0, -h)) \\
&\qquad\qquad\qquad \cdot \overline{b_{h_i}(zu, zuv, zu\gamma_i(v)^{-1}v)} \\
&= (\Delta_{\psi(2\ell h_i h, -h)} b_{h_i})(zu, zuv, zu\gamma_i(v)^{-1}v),
\end{aligned}$$

and so we can re-express the above coboundary equation as

$$\tau_i(zu, zuv, zu\gamma_i(v)^{-1}v) = \Delta_{\psi(2\ell h_i h, -h)} b'_i(zu, zuv, zv).$$

On the other hand, recalling the consequences of the directional CL-structure obtained in Corollary 3.19, we know that there are Borel maps $b_i^\circ : K_{(2\ell h_i, 0)} \times Z \to \mathrm{S}^1$ and $c_i^\circ : K_{(2\ell h_i, 0)} \times Z/\overline{\phi(h\mathbb{Z}^2)} \to \mathrm{S}^1$ such that

$$\begin{aligned}
&\overline{\tau_i(zu, zuv, zv)} \cdot \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v) \\
&= \Delta_{u\gamma_i(v)^{-1}} \overline{\chi_1} \circ \sigma(-h\mathbf{e}_2, zv) \\
&= b_i^\circ(u\gamma_i(v)^{-1}, zv \cdot \phi(0, -h)) \cdot \overline{b_i^\circ(u\gamma_i(v)^{-1}, zv)} \cdot c_i^\circ(u\gamma_i(v)^{-1}, zv \cdot \overline{\phi(h\mathbb{Z}^2)}).
\end{aligned}$$

Moreover, recalling that we have reduced to the case in which $h\mathbb{Z}^2$ acts ergodically through R_ϕ, the dependence on the coset $zv \cdot \overline{\phi(h\mathbb{Z}^2)}$ above may be dropped.

Since the map $(zu, zuv, zv) \mapsto (u\gamma_i(v)^{-1}, zv)$ is also easily seen to be a homomorphism onto a finite-index (and hence positive-measure) subgroup of $K_{(2\ell h_i,0)} \times Z$, the above holds $m_{\tilde{Z}_0}$-almost everywhere. In addition, if we now define

$$b_i''(zu, zuv, zv) := b_i^\circ(u\gamma_i(v)^{-1}, zv)$$

then using again that fact that $\phi(2\ell h_i h, 0)\gamma(\phi(0, -h))^{-1} = 1$ we can compute directly that

$$\begin{aligned}&\Delta_{\psi(2\ell h_i h,-h)} b_i''(zu, zuv, zv)\\&= b_i''(zu \cdot \phi(2\ell h_i h, 0), zuv \cdot \phi(2\ell h_i h, -h), zv \cdot \phi(0, -h)) \cdot \overline{b_i''(zu, zuv, zv)}\\&= b_i^\circ(u\gamma_i(v)^{-1}, zv \cdot \phi(0, -h)) \cdot \overline{b_i^\circ(u\gamma_i(v)^{-1}, zv)},\end{aligned}$$

and so we can re-express the above coboundary equation as

$$\begin{aligned}&\overline{\tau_i(zu, zuv, zv)} \cdot \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v)\\&\qquad\qquad = \Delta_{\psi(2\ell h_i h,-h)} b_i''(zu, zuv, zv) \cdot c_i^\circ(u\gamma_i(v)^{-1}).\end{aligned}$$

Finally we can put the coboundary equations obtained above together by setting $\tilde{b}_i := b_i' \cdot \overline{b_i''}$ and

$$\tilde{c}_i(u, v) := \overline{c_i^\circ(u\gamma_i(v)^{-1})}$$

to obtain

$$\begin{aligned}&\tau_i(zu, zuv, zv)\\&= \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v) \cdot \overline{\left(\overline{\tau_i(zu, zuv, zv)} \cdot \tau_i(zu, zuv, zu\gamma_i(v)^{-1}v)\right)}\\&= \Delta_{\psi(2\ell h_1 h,-h)} \tilde{b}_i(zu, zuv, zv) \cdot \tilde{c}_i(u, v)\end{aligned}$$

$m_{\tilde{Z}_0}$-almost everywhere, where $\tilde{c}_i(u, v)$ is $R_{(\phi(2\ell h_1 h,0),\phi(0,-h))}$-invariant, as required. □

The remaining steps in the proof of Proposition 3.25 follow quite closely the ideas of Host and Kra's neat approach in [16] to the convergence of triple linear averages associated to three powers of a single ergodic transformation.

The main technical result we need is the 'compactification' result for the family of functions

$$\chi_1 \circ \sigma(n_1\mathbf{e}_1, \,\cdot\,) \cdot \chi_2 \circ \sigma(n_1\mathbf{e}_1 + n_2\mathbf{e}_2, \,\cdot\,) \cdot \overline{\chi_2} \circ \sigma(n_2\mathbf{e}_2, \,\cdot\,) \qquad (n_1, n_2) \in \Gamma$$

given in the next proposition. This will serve as our analog of Lemma 4.2 of [16], but it differs from that result in certain important details. Most notably, our proposition is a little more 'quantitative', as a result of the introduction of an additional 'phase function' given by a generalized polynomial. Generalized polynomials have been objects of interest among ergodic theorists for some time, and so we recall their definition here for completeness but will refer elsewhere for their properties that we need.

Definition 3.30 (Gen-polynomials)**.** *A map $p : \mathbb{Z}^2 \to \mathbb{R}$ is a* ***generalized polynomial ('gen-polynomial')*** *if it can be expressed using repeated composition of ordinary real-valued polynomials and the operations of taking the integer part, addition and multiplication.*

For the basic properties of gen-polynomials we refer to Bergelson and Leibman [8], Leibman [20] and the references given there. Recall that we have now restricted our attention to the subgroup $\Gamma := \mathbb{Z}(2\ell h_1 h, h) + \mathbb{Z}(2\ell h_2 h, h)$, and let us henceforth write $\mathbf{q}_i = (q_{i1}, q_{i2}) := (2\ell h_i h, h)$ for brevity. We also now abbreviate $K_1 := K_{(h,0)}$ and $K_2 := K_{(0,h)}$, and observe from the DIO property that $K_1 \cdot K_2 \cong K_1 \times K_2$ in Z, so in particular for any $\mathbf{n} \in \Gamma \leq h\mathbb{Z}^2$ we may interpret each $\phi(\mathbf{n})$ uniquely as a member of $K_1 \times K_2$.

Proposition 3.31. *There is a gen-polynomial $p : \mathbb{Z}^2 \to \mathbb{R}$ for which the following holds. For any $\alpha > 0$ there are*

- *a Borel function $C_\alpha : Z \times (K_1 \times K_2)^2 \to \mathrm{S}^1$ such that the family of slices*
$$Z \mapsto \mathrm{S}^1 : z \mapsto C_\alpha(z, u_1, v_1, u_2, v_2)$$
indexed by $(u_1, v_1, u_2, v_2) \in (K_1 \times K_2)^2$ all lie in $L^2(m_Z)$ and vary continuously with (u_1, v_1, u_2, v_2), and
- *an open subset $U_\alpha \subseteq K_1 \times K_2$ of the form*
$$U_\alpha = \bigcap_{\gamma \in \mathcal{F}} \{(u, v) \in K_1 \times K_2 : \ \delta < \{\gamma(u, v)\} < 1 - \delta\}$$
for some $\delta > 0$ and some finite subset $\mathcal{F} \subseteq \widehat{K_1 \times K_2}$ such that $\gamma(\phi(\mathbf{q}_1)) \in \mathrm{S}^1$ is irrational for every $\gamma \in \mathcal{F}$ and $m_{K_1 \times K_2}(U_\alpha) > 1 - \alpha$

such that

$$\begin{aligned} \exp(2\pi \mathrm{i} p(m, n)) \cdot \chi_2 \circ \sigma((mq_{11} + nq_{21}, 0), z) \\ \cdot \chi_1 \circ \sigma((mq_{11} + nq_{21}, mq_{12} + nq_{22}), z) \cdot \overline{\chi_1} \circ \sigma((0, mq_{12} + nq_{22}), z) \\ = C_\alpha(z, \phi(m\mathbf{q}_1), \phi(n\mathbf{q}_2)) \qquad \textit{for Haar-a.e. } z \in Z \end{aligned}$$

for every $m \in \mathbb{Z}$ such that $\phi(m\mathbf{q}_1) \in U_\alpha$, where we use our identification of $\phi(m\mathbf{q}_i) \in K_1 \cdot K_2 \leq Z$ with a member of $K_1 \times K_2$.

We will prove this proposition following a couple of preparatory lemmas. The first of these is a simple calculation from Lemma 3.29.

Lemma 3.32. *For any point $m\mathbf{q}_1 + n\mathbf{q}_2 \in \Gamma$ we have*

$$\begin{aligned}
&\chi_2 \circ \sigma((2\ell h_1 hm + 2\ell h_2 hn, 0), zu) \\
&\qquad \cdot \chi_1 \circ \sigma((2\ell h_1 hm + 2\ell h_2 hn, -hm - hn), zuv) \\
&\qquad\qquad \cdot \overline{\chi_1} \circ \sigma((0, -hm - hn), zu) \\
&= \Delta_{\psi(2\ell h_1 hm, -hm)} \tilde{b}_1(zu, zuv, zv) \\
&\qquad\qquad \cdot \Delta_{\psi(2\ell h_2 hn, -hn)} \tilde{b}_2(zu\phi(2\ell h_1 hm, 0), zuv\phi(2\ell h_1 hm, -hm), zv\phi(-hm)) \\
&\qquad\qquad \cdot \tilde{c}_1(u, v)^m \cdot \tilde{c}_2(u\phi(2\ell h_1 hm, 0), v\phi(0, -hm))^n.
\end{aligned}$$

Proof This follows immediately from the separate conclusions of Lemma 3.29 for $i = 1$ and $i = 2$ by observing the consequences of the defining equations for a cocycle over a $\mathbb{Z}^2$-action that

$$\begin{aligned}
&\sigma((2\ell h_1 hm + 2\ell h_2 hn, 0), zu) \\
&\qquad\qquad = \sigma((2\ell h_1 hm, 0), zu) \cdot \sigma((2\ell h_2 hn, 0), zu \cdot \phi(2\ell h_1 hm, 0)),
\end{aligned}$$

$$\begin{aligned}
&\sigma((2\ell h_1 hm + 2\ell h_2 hn, -hm - hn), zuv) \\
&\quad = \sigma((2\ell h_1 hm, -hm), zuv) \cdot \sigma((2\ell h_2 hn, -hn), zuv \cdot \phi(2\ell h_1 hm, -hm)),
\end{aligned}$$

and

$$\sigma((-hm - hn), zv) = \sigma((0, -hm), zv) \cdot \sigma((0, -hn), zv \cdot \phi(0, -hm)),$$

and then multiplying these together. $\square$

The proof of Proposition 3.31 will also require the following analog of an enabling lemma from Host and Kra [16].

Lemma 3.33 (C.f. Lemma 3.3 in [16])**.** *Suppose that $\delta < 1/100$ and that $f_i : Z \to \mathrm{S}^1$, $i = 1, 2, 3$, and $h : K_1 \times K_2 \to \mathrm{S}^1$ are Borel functions such that*

$$f_1(zu) f_2(zuv) f_3(zv) h(u, v) \approx_\delta 1 \qquad \text{in } L^2(m_{Z \times K_1 \times K_2})$$

Then there are Borel functions $g_1 : Z/K_1 \to \mathrm{S}^1$ and $g_2 : Z/K_2 \to \mathrm{S}^1$, characters $\gamma_1 \in K_2^\perp$ and $\gamma_2 \in K_1^\perp$ and constants $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{S}^1$ and $\beta \in \mathrm{S}^1$ satisfying $\alpha_1\alpha_2\alpha_3\beta = 1$ such that

$$f_1(z) \approx_{\mathrm{o}_\delta(1)} \alpha_1 g_1(zK_1)\gamma_2(z),$$

$$f_2(z) \approx_{\mathrm{o}_\delta(1)} \alpha_2 \overline{g_1(zK_1)\gamma_1(z)g_2(zK_2)\gamma_2(z)},$$

$$f_3(z) \approx_{\mathrm{o}_\delta(1)} \alpha_3 g_2(zK_2)\gamma_1(z)$$

and

$$h(u,v) \approx_{\mathrm{o}_\delta(1)} \beta\gamma_1(u)\gamma_2(v),$$

where all approximations hold in the norm of the relevant L^2 space. Consequently we also have

$$\begin{aligned} &f_1(z)f_2(z)f_3(z) \\ \approx_{\mathrm{o}_\delta(1)}\ &\alpha_1 g_1(zK_1)\chi_2(z)\cdot\alpha_2\overline{g_1(zK_1)\gamma_1(z)g_2(zK_2)\gamma_2(z)}\cdot\alpha_3 g_2(zK_2)\gamma_2(z) \equiv \beta \end{aligned}$$

in $L^2(m_Z)$.

Proof Recalling that the system (Z, m_Z, ϕ) is DIO and that $K_1 \cdot K_2$ has finite index in Z, by restriction to a coset we may assume that $Z = K_1 \times K_2$, and so write the given equation as

$$f_1(z_1u, z_2)f_2(z_1u, z_2v)f_3(z_1, z_2v)h(u,v) \approx_\delta 1 \qquad \text{in } L^2(m_{K_1\times K_2\times K_1\times K_2}).$$

In the argument below all approximations $\approx$ will implicitly refer to an error of the form $\mathrm{o}_\delta(1)$.

Changing variables so that $z_1' := z_1u$ and $v' := z_2v$, this becomes

$$f_1(z_1', z_2)f_2(z_1', v')f_3(z_1'u^{-1}, v')h(u, v'z_2^{-1}) \approx_\delta 1 \qquad \text{in } L^2(m_{K_1\times K_2\times K_1\times K_2}),$$

and so for most fixed choices of u and v' we have

$$f_1(z_1', z_2) \approx_\delta \overline{f_2(z_1', v')f_3(z_1'u^{-1}, v')h(u, v'z_2^{-1})} \qquad \text{in } L^2(m_{K_1\times K_2}),$$

which is manifestly a product of functions each of which depends only on z_1' (or, equivalently, on z_1) or only on z_2. We may therefore approximate

$$f_1(z_1, z_2) \approx g_{11}(z_1)g_{12}(z_2)$$

for some $g_{1i} : K_i \to \mathrm{S}^1$, and exactly similarly we can approximate

$$f_3(z_1, z_2) \approx g_{31}(z_1)g_{32}(z_2).$$

Substituting these right-hand sides into our original approximation we obtain

$$g_{11}(z_1u)g_{13}(z_2)f_2(z_1u, z_2v)g_{31}(z_1)g_{32}(z_2v)h(u,v) \approx 1 \qquad \text{in } L^2(m_{K_1\times K_2\times K_1\times K_2}),$$

or, changing variables to $z_1' := z_1u$ and $z_2' := z_2v$,

$$g_{11}(z_1')g_{13}(z_2'v^{-1})f_2(z_1', z_2')g_{31}(z_1u^{-1})g_{32}(z_2')h(u,v) \approx 1 \qquad \text{in } L^2(m_{K_1\times K_2\times K_1\times K_2}).$$

Again fixing some u and v for which this is true for most z_1' and z_2', we find that f_2 must also take an approximate product form,

$$f_2(z_1, z_2) \approx g_{21}(z_1)g_{22}(z_2),$$

while fixing instead z_1' and z_2' and allowing u and v to vary we obtain the same conclusion for h:

$$h(u,v) \approx h_1(u)h_2(v).$$

Now we substitute all these approximate factorizations back into our original approximation one last time to obtain

$$\big((g_{11}\cdot g_{21})(z_1u)\cdot g_{31}(z_1)\cdot h_1(u)\big)\cdot\big(g_{12}(z_2)\cdot(g_{22}\cdot g_{32})(z_2v)\cdot h_2(v)\big) \approx 1 \qquad \text{in } L^2(m_{K_1\times K_2\times K_1\times K_2}),$$

and so in fact we must have that $\big((g_{11}\cdot g_{21})(z_1u)\cdot g_{31}(z_1)\cdot h_1(u)\big)$ is close to a constant-valued map in $L^2(m_{K_1\times K_1})$ and similarly that $\big(g_{12}(z_2)\cdot(g_{22}\cdot g_{32})(z_2v)\cdot h_2(v)\big)$ is close in $L^2(m_{K_2\times K_2})$ to a map with value the inverse of that constant.

Calling this constant $\gamma \in \mathrm{S}^1$ and writing $h_1' := \gamma\cdot h_1$, we are left with the approximate equation

$$(g_{11}\cdot g_{21})(z_1u)\cdot g_{31}(z_1)\cdot h_1'(u) \approx 1$$

in $L^2(m_{K_1\times K_1})$. Since the functions $g_{11}\cdot g_{21}$, g_{31} and h_1' take values in S^1, they all have norm 1 in $L^2(m_{K_1})$. On the other hand, averaging over z_1 in the above approximation gives that

$$\overline{h_1'} \approx (g_{11}\cdot g_{31}) * g_{31}'$$

where we define $g_{31}'(z_1) := \overline{g_{31}(z_1^{-1})}$. Hence, taking the Fourier transform of this approximation gives

$$\|\overline{h_1'} - (g_{11}\cdot g_{31}) * g_{31}'\|_2^2 = \sum_{\gamma\in\widehat{K_1}} |\widehat{\overline{h_1'}}(\gamma) - (\widehat{g_{11}\cdot g_{31}})(\gamma)\cdot\widehat{g_{31}'}(\gamma)|^2 \approx 0$$

and hence also

$$1 = \|h_1'\|_2^2 = \sum_{\gamma\in\widehat{K_1}} |\widehat{h_1'}(\gamma)|^2 \approx \sum_{\gamma\in\widehat{K_1}} |(\widehat{g_{11}\cdot g_{31}})(\gamma)\cdot\widehat{g_{31}'}(\gamma)|^2.$$

On the other hand, simply by the non-negativity of all the terms involved we have

$$\sum_{\gamma\in\widehat{K_1}} |(\widehat{g_{11}\cdot g_{31}})(\gamma)|^2|\widehat{g_{31}'}(\gamma)|^2 \leq \Big(\sum_{\gamma\in\widehat{K_1}} |(\widehat{g_{11}\cdot g_{31}})(\gamma)|^2\Big)\Big(\sum_{\gamma\in\widehat{K_1}} |\widehat{g_{31}'}(\gamma)|^2\Big) \leq 1$$

with approximate equality only if $\widehat{g_{31}'}$ and $(\widehat{g_{11}\cdot g_{31}})$ are both concentrated on a single character.

Thus the above approximation in $L^2(m_{K_1\times K_1})$ is possible only if there are some character $\gamma_1 \in \widehat{K_1}$ and some constants $\alpha_3,\eta_1 \in \mathrm{S}^1$ such that $g_{31} \approx \alpha_3\gamma_1$, $g_{11}\cdot g_{31} \approx \eta_1\overline{\gamma_1}$ and $h_1' \approx \overline{\alpha_3\eta_1}\gamma_1$. Exactly similarly we obtain a character $\gamma_2 \in \widehat{K_2}$ and constants $\alpha_1,\eta_2 \in \mathrm{S}^1$ such that $g_{12} \approx \alpha_1\gamma_2$, $g_{22}\cdot g_{32} \approx \eta_2\overline{\gamma_2}$ and $h_2' \approx \overline{\alpha_1\eta_2}\gamma_2$. Setting $\alpha_2 := \overline{\eta_1\eta_2}$ and $\beta := \overline{\alpha_1\eta_2}\cdot\overline{\alpha_3\eta_1} = \overline{\alpha_1\alpha_2\alpha_3}$, we see that combining these resulting approximants gives the result. The final assertion that

$$\begin{aligned}&f_1(z)f_2(z)f_3(z)\\ \approx_{\mathrm{o}_\delta(1)}\ &\alpha_1 g_1(zK_1)\gamma_2(z)\cdot\alpha_2\overline{g_1(zK_1)\gamma_1(z)g_2(zK_2)\gamma_2(z)}\cdot\alpha_3 g_2(zK_2)\gamma_2(z) \equiv \beta\end{aligned}$$

in $L^2(m_Z)$ follows immediately. □

Proof of Proposition 3.31 This will rest on the special form of the functions $\tilde{c}_i$ obtained from Proposition 2.1 and its consequence Lemma 3.29. Those results tell us that these functions are of the form

$$\tilde{c}_i(u,v) = \alpha_i(u,v)\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_i} a_{i,j}(u,v)\{\chi_{i,j}(\phi(2\ell h_i h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{i,j}(u,v)\}\Big)$$

for some maps $\alpha_1,\alpha_2 : K_1\times K_2 \in \mathrm{S}^1$ and $a_{i,j} : K_1\times K_2 \to \mathbb{Z}$ that factorize through some finite quotient group and some characters $\gamma_{i,j},\chi_{i,j} \in \widehat{K_1\times K_2}$.

In this expression, we note that if for some j the character $\gamma_{i,j}$ has image a finite subgroup of S^1, rather than the whole of S^1, then we can simply replace $\alpha_i(u,v)$ by

$$\alpha_i(u,v)\cdot\exp(2\pi\mathrm{i}a_{i,j}(u,v)\{\chi_{i,j}(\phi(2\ell h_i h\mathbf{e}_1),\phi(h\mathbf{e}_2))\}\{\gamma_{i,j}(u,v)\})$$

and remove the term $a_{i,j}(u,v)\{\chi_{i,j}(\phi(2\ell h_i r\mathbf{e}_1),\phi(r\mathbf{e}_2))\}\{\gamma_{i,j}(u,v)\}$ from the sum inside the main exponential. Therefore we may assume further that in this expression the characters $\gamma_{i,j}$ all map $K_1\times K_2$ onto the whole of S^1. Having made these

arrangements, we may now choose some large integer $r \geq 1$ for which each α_i and ai, j is actually constant on each coset of $\overline{\psi(r\Gamma)}$. Replacing h with rh, each $\mathbf{q}_i$ with $r\mathbf{q}_i$, and thus Γ with the further finite-index sublattice $r\Gamma \leq \Gamma$, we may now simply assume that each α_i and $a_{i,j}$ is constant.

Now let $\mathcal{J} \subseteq \{1, 2, \ldots, J_2\}$ be the subset of indices for which $\gamma_{2,j}(\mathbf{q}_1)$ is an irrational element of the circle group S^1. From the condition that each $\gamma_{i,j}$ have range equal to the whole of S^1 it follows that for any $\alpha > 0$ there is some $\delta(\alpha) > 0$ such that the open set

$$U_\alpha := \{(u, v) \in K_1 \times K_2 : \ \delta(\alpha) < \{\gamma_{2,j}(u, v)\} < 1 - \delta(\alpha) \ \forall j \in \mathcal{J}\}$$

has $m_{K_1\times K_2}(U_\alpha) > 1 - \alpha$. In addition, we may take $\alpha \mapsto \delta(\alpha)$ to be strictly increasing for sufficiently small α, so that $\overline{U_\alpha} \subseteq U_{\alpha/2}$. We will obtain the function C_α by showing that for a suitably chosen generalized polynomial p, for any sequence $(m_k, n_k)_{k\geq 1}$ in $\mathbb{Z}^2$ such that

$$\phi(m_k\mathbf{q}_1) \in U_{\alpha/2} \qquad \forall k,$$

$$\phi(m_k\mathbf{q}_1) \to (u_1^\circ, v_1^\circ) \in K_1 \times K_2 \qquad \text{as } k \to \infty$$

and

$$\phi(n_k\mathbf{q}_2) \to (u_2^\circ, v_2^\circ) \in K_1 \times K_2 \qquad \text{as } k \to \infty,$$

we have that the sequence of functions

$$\begin{aligned} &z \mapsto \exp(2\pi \mathrm{i} p(m_k, n_k)) \cdot \chi_2 \circ \sigma(((m_k q_{11} + n_k q_{21}, 0), z) \\ &\cdot \chi_1 \circ \sigma((m_k q_{11} + n_k q_{21}, m_k q_{12} + n_k q_{22}), z) \cdot \overline{\chi_1} \circ \sigma((0, m_k q_{21} + n_k q_{22}), z) \end{aligned}$$

on Z converges in $L^2(m_Z)$. From this it follows that for any $(u_1, v_1) \in U_{\alpha/2}$ we may unambiguously define a function $z \mapsto C'_\alpha(z, u_1, v_1, u_2, v_2)$ to be the limit of these functions when $u_i = u_i^\circ$ and $v_i = v_i^\circ$, and this defines a Borel map C'_α on $Z \times U_{\alpha/2} \times (K_1 \times K_2)$ such that $(u_1, v_1, u_2, v_2) \mapsto C'_\alpha(\,\cdot\,, u_1, v_1, u_2, v_2)$ is a continuous map from $U_{\alpha/2} \times (K_1 \times K_2)$ to $L^2(m_Z)$. Having done this we can simply choose any continuous function φ satisfying $1_{U_\alpha} \leq \varphi \leq 1_{U_{\alpha/2}}$ and define

$$C_\alpha(z, u_1, v_1, u_2, v_2) := \begin{cases} \varphi(u_1, v_1) C'_\alpha(z, u_1, v_1, u_2, v_2) & \text{if } (u_1, v_1) \in U_{\alpha/2} \\ 0 & \text{else:} \end{cases}$$

it is now clear that this function has the desired properties in conjunction with the set U_α.

Thus it remains to show this convergence for an arbitrary such sequence (m_k, n_k). Letting

$$\begin{aligned} f_{1,k}(z) &:= \chi_2 \circ \sigma((m_k q_{11} + n_k q_{21}, 0), z), \\ f_{2,k}(z) &:= \chi_1 \circ \sigma((m_k q_{11} + n_k q_{21}, m_k q_{12} + n_k q_{22}), z) \\ \text{and } f_{3,k} &:= \overline{\chi_1} \circ \sigma((0, m_k q_{21} + n_k q_{22}), z), \end{aligned}$$

from Lemma 3.32 we have

$$\begin{aligned} & f_{1,k}(zu) f_{2,k}(zuv) f_{2,k}(zv) \\ & = \Delta_{\psi(2\ell h_1 h m_k, -h m_k)} \tilde{b}_1(zu, zuv, zv) \\ & \qquad \cdot \Delta_{\psi(2\ell h_2 h n_k, -h n_k)} \tilde{b}_2(zu\phi(2\ell h_1 h m_k, 0), zuv\phi(2\ell h_1 h m_k, -h m_k), zv\phi(-h m_k)) \\ & \qquad \cdot \tilde{c}_1(u,v)^{m_k} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_k, 0), v\phi(0, -h m_k))^{n_k}. \end{aligned}$$

Re-arranging, we deduce that

$$\begin{aligned} & f_{1,k}(zu) f_{2,k}(zuv) f_{2,k}(zv) \cdot \left(\tilde{c}_1(u,v)^{m_k} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_k, 0), v\phi(0, -h m_k))^{n_k}\right) \\ & = \Delta_{\psi(2\ell h_1 h m_k, -h m_k)} \tilde{b}_1(zu, zuv, zv) \\ & \qquad \cdot \Delta_{\psi(2\ell h_2 h n_k, -h n_k)} \tilde{b}_2(zu\phi(2\ell h_1 h m_k, 0), zuv\phi(2\ell h_1 h m_k, -h m_k), zv\phi(-h m_k)) \\ & \to \Delta_{(u_1^\circ, u_1^\circ v_1^\circ, v_1^\circ)} \tilde{b}_1(zu, zuv, zv) \cdot \Delta_{(u_2^\circ, u_2^\circ v_2^\circ, v_2^\circ)} \tilde{b}_2(zuu_{11}, zuu_1^\circ v v_1^\circ, zvv_1^\circ) \end{aligned}$$

in $L^2(m_{Z_0})$ as $k \to \infty$, and hence that

$$\begin{aligned} & f_{1,k}(zu)\overline{f_{1,\ell}(zu)} f_{2,k}(zuv)\overline{f_{2,\ell}(zuv)} f_{3,k}(zv)\overline{f_{3,\ell}(zv)} \\ & \qquad \cdot \tilde{c}_1(u,v)^{m_k} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_k, 0), v\phi(0, -h m_k))^{n_k} \\ & \qquad \overline{\cdot \tilde{c}_1(u,v)^{m_\ell} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_\ell, 0), v\phi(0, -h m_\ell))^{n_\ell}} \\ & \to 0 \end{aligned}$$

in $L^2(m_{Z_0})$ as $k, \ell \to \infty$.

It now follows from Lemma 3.33 that in $L^2(m_Z)$ the S^1-valued function

$$f_{1,k}(z)\overline{f_{1,\ell}(z)} f_{2,k}(z)\overline{f_{2,\ell}(z)} f_{3,k}(z)\overline{f_{3,\ell}(z)}$$

approaches the subset of constant S^1-valued functions in $L^2(m_Z)$ as $k, \ell \to \infty$, and that $\beta_{k,\ell} \in \mathrm{S}^1$ is a family of constants to which the above functions are asymptotically equal if and only if the function

$$\begin{aligned} & \beta_{k,\ell} \cdot \tilde{c}_1(u,v)^{m_k} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_k, 0), v\phi(0, -h m_k))^{n_k} \\ & \qquad \cdot \overline{\tilde{c}_1(u,v)^{m_\ell} \cdot \tilde{c}_2(u\phi(2\ell h_1 h m_\ell, 0), v\phi(0, -h m_\ell))^{n_\ell}} \end{aligned}$$

is close in $L^2(m_{K_1\times K_2})$ to a character (which is necessarily unique once this approximation is sufficiently good, since all distinct characters are separated by a distance of $\sqrt{2}$ in $L^2(m_{K_1\times K_2})$).

To complete the proof, it will therefore suffice to find some gen-polynomial $p(m,n)$ (not depending on the choices we made above for a particular α) such that the constants $\beta_{k,\ell} = \exp(2\pi\mathrm{i}(p(m_k,n_k) - p(m_\ell,n_\ell)))$ satisfy this latter condition. We will now see that such a gen-polynomial can simply be read off from the special form of the functions $\tilde{c}_1$ and $\tilde{c}_2$ guaranteed by Lemma 3.29 and recalled above.

Indeed, having replaced Γ with the sufficiently small finite-index subgroup $r\Gamma$ and re-assigned our notation, these functions are of the form

$$\tilde{c}_i(u,v) = \alpha_i \exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_i} a_{i,j}\{\chi_{i,j}(\phi(2\ell h_i h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{i,j}(u,v)\}\Big)$$

and for some $\alpha_1,\alpha_2\in\mathrm{S}^1$, $a_{i,j}\in\mathbb{Z}$ and characters $\gamma_{i,j},\chi_{i,j}\in\widehat{K_1\times K_2}$ whose images are the whole circle group S^1. In terms of these expressions we can now write

$$\begin{aligned}
&\tilde{c}_1(u,v)^{m_k}\cdot\tilde{c}_2(u\phi(2\ell h_1 h m_k,0),v\phi(0,-hm_k))^{n_k}\\
&=\alpha_1^{m_k}\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_1} m_k a_{1,j}\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{1,j}(u,v)\}\Big)\\
&\quad\cdot\alpha_2^{n_k}\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_2} n_k a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\\
&\qquad\qquad\qquad\qquad\cdot\{\gamma_{2,j}(u\phi(2\ell h_1 h m_k\mathbf{e}_1),v\phi(-hm_k\mathbf{e}_2))\}\Big).
\end{aligned}$$

In order to use this expression we next note the elementary identity

$$\begin{aligned}
&\{\gamma_{2,j}(u\phi(2\ell h_1 h m_k\mathbf{e}_1),v\phi(-hm_k\mathbf{e}_2))\}\\
&=\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\\
&\quad-\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor.
\end{aligned}$$

Substituting this identity and its partner for (m_ℓ,n_ℓ) and taking the difference of

the results we obtain

$$\begin{aligned}
&\tilde{c}_1(u,v)^{m_k}\cdot\tilde{c}_2(u\phi(2\ell h_1 h m_k,0),v\phi(0,-hm_k))^{n_k}\\
&\qquad\qquad\qquad\qquad\cdot\overline{\tilde{c}_1(u,v)^{m_\ell}\cdot\tilde{c}_2(u\phi(2\ell h_1 h m_\ell,0),v\phi(0,-hm_\ell))^{n_\ell}}\\
&=\alpha_1^{m_k-m_\ell}\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_1}a_{1,j}(m_k-m_\ell)\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{1,j}(u,v)\}\Big)\\
&\quad\cdot\alpha_2^{n_k-n_\ell}\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_2}a_{2,j}(n_k-n_\ell)\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{2,j}(u,v)\}\Big)\\
&\quad\cdot\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_2}a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\big(n_k\{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\\
&\qquad\qquad\qquad\qquad-n_\ell\{\gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}\big)\Big)\\
&\quad\cdot\exp\Big(-2\pi\mathrm{i}\sum_{j=1}^{J_2}a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\\
&\qquad\qquad\cdot\big(n_k\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor\\
&\qquad\qquad\quad-n_\ell\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}\rfloor\big)\Big).
\end{aligned}$$

Let us now consider some of the factors in this product in turn.

- First, we have by assumption that $\phi(2\ell h_1 h m_k\mathbf{e}_1)\to u_1^\circ$ and $\phi(-hm_k\mathbf{e}_2)\to v_1^\circ$ as $k\to\infty$. Since $\chi_{1,j}$ is a character on $K_1\times K_2$, it follows that

  $$\mathrm{dist}\big((m_k-m_\ell)\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\,,\,\mathbb{Z}\big)\to 0$$

  as $k,\ell\to\infty$. Let us here write $I(r)\in\mathbb{Z}$ for the closest integer to any $r\in\mathbb{R}$, rounding down when r is a proper half-integer, so that $I(r)\in\{\lfloor r\rfloor,\lfloor r\rfloor+1\}$. From the above it follows that as $k,\ell\to\infty$ the distance in $L^2(m_{K_1\times K_2})$ between the function

  $$(u,v)\mapsto\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_1}a_{1,j}(m_k-m_\ell)\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\{\gamma_{1,j}(u,v)\}\Big)$$

and the character

$$\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_1} a_{1,j} I\big((m_k - m_\ell)\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1), \phi(-h\mathbf{e}_2))\}\big)\{\gamma_{1,j}(u,v)\}\Big)$$
$$= \prod_{j=1}^{J_1} \gamma_{1,j}(u,v)^{a_{1,j} I\big((m_k - m_\ell)\{\chi_{1,j}(\phi(2\ell h_1 h\mathbf{e}_1), \phi(-h\mathbf{e}_2))\}\big)}$$

tends to 0. Exactly similarly the functions

$$\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_2} a_{2,j}(n_k - n_\ell)\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1), \phi(-h\mathbf{e}_2)\}\{\gamma_{2,j}(u,v)\}\Big)$$

are also asymptotically close to characters as $k, \ell \to \infty$, and hence the same is true of the product of these two exponential functions.

- Now consider the last factor above,

$$\begin{aligned}
&\exp\Big(-2\pi\mathrm{i}\sum_{j=1}^{J_2} a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1), \phi(-h\mathbf{e}_2)\} \\
&\qquad\qquad \cdot\big(n_k\lfloor\{\gamma_{2,j}(u,v)\} + \{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1), \phi(-h m_k\mathbf{e}_2))\}\rfloor \\
&\qquad\qquad\qquad - n_\ell\lfloor\{\gamma_{2,j}(u,v)\} + \{\gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1), \phi(-h m_\ell\mathbf{e}_2))\}\rfloor\big)\Big) \\
&= \prod_{j=1}^{J_2}\exp\Big(-2\pi\mathrm{i} a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2 h\mathbf{e}_1), \phi(-h\mathbf{e}_2)\} \\
&\qquad\qquad \cdot\big(n_k\lfloor\{\gamma_{2,j}(u,v)\} + \{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1), \phi(-h m_k\mathbf{e}_2))\}\rfloor \\
&\qquad\qquad\qquad - n_\ell\lfloor\{\gamma_{2,j}(u,v)\} + \{\gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1), \phi(-h m_\ell\mathbf{e}_2))\}\rfloor\big)\Big)
\end{aligned}$$

We will argue that each of the individual factors of this product over j is asymptotically close to the constant function 1 in $L^2(m_{K_1\times K_2})$, using again the fact that

$$\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1), \phi(-h m_k\mathbf{e}_2))\,,\ \gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1), \phi(-h m_\ell\mathbf{e}_2)) \to \gamma_{2,j}(u_1^\circ, v_1^\circ)$$

as $k, \ell \to \infty$. For this argument we must treat the cases $j \in \mathcal{J}$ and $j \notin \mathcal{J}$ separately.

If $j \in \mathcal{J}$, then we know that $\delta(\alpha/2) \leq \{\gamma_{2,j}(u_1^\circ, v_1^\circ)\} \leq 1 - \delta(\alpha/2)$ from the restriction $\phi(m_k\mathbf{q}_1) \in U_{\alpha/2}$ and continuity. This implies that once k and ℓ are sufficiently large then we have that

$$\{\gamma_{2,j}(\phi(2\ell h_1 h m_k\mathbf{e}_1), \phi(-h m_k\mathbf{e}_2))\} \quad \text{and} \quad \{\gamma_{2,j}(\phi(2\ell h_1 h m_\ell\mathbf{e}_1), \phi(-h m_\ell\mathbf{e}_2))\}$$

lie close together and both inside $(0,1)$. From this we deduce that

$$\begin{aligned} m_{K_1\times K_2}\big\{(u,v)\in K_1\times K_2:\ &\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor\\ &\neq\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}\rfloor\big\}\\ \to 0\qquad\qquad& \end{aligned}$$

as $k,\ell\to\infty$, and so in this case the j^{th} function in the above product is asymptotically close in $L^2(m_{K_1\times K_2})$ to the function

$$\begin{aligned} &\exp\big(-2\pi\mathrm{i}a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2)\}\\ &\qquad\qquad\cdot(n_k-n_\ell)\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor\big)\\ &=\chi_{2,j}(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2))^{a_{2,j}(n_\ell-n_k)\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor}, \end{aligned}$$

and this is close to 1 for either of the possible values (0 or 1) of $\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor$, because $a_{2,j}$ is a fixed integer and

$$(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2))^{n_k}\approx(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2))^{n_\ell}$$

when k and ℓ are large.

On the other hand, if $j\in\{1,2,\dots,J_2\}\setminus\mathcal{J}$ then $\gamma_{2,j}(\phi(2\ell h_1h,-h))$ is a root of unity, and so since the sequence $\phi(m_k\mathbf{q}_1)$ converges the values $\gamma_{2,j}(\phi(2\ell h_1hm_k,-hm_k))$ are eventually constant. Once this is so, of course we have

$$\begin{aligned} &\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\rfloor\\ &\qquad\qquad=\lfloor\{\gamma_{2,j}(u,v)\}+\{\gamma_{2,j}(\phi(2\ell h_1hm_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}\rfloor, \end{aligned}$$

for all $(u,v)\in K_1\times K_2$ and we may complete the proof of this case as above.

Remark It is for the above argument that we must make a restriction such as $\phi(m_k\mathbf{q}_1)\in U_{\alpha/2}$. Indeed, without this we might have chosen a limit point (u_1°,v_1°) for which $\gamma_{2,j}(u_1^\circ,v_1^\circ)=0$ for some $j\in\mathcal{J}$, and in this case it will generally happen that there are large k and ℓ for which, say, $\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}$ is very slightly more than 0 but $\{\gamma_{2,j}(\phi(2\ell h_1hm_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}$ is very slightly less than 1. This disrupts the above argument that the last factor in our large product is close to 1, and we find instead that it might be close to some other constant, which seems to be hard to account for in the desired expression $p(m_k,n_k)-p(m_\ell,n_\ell)$. $\lhd$

Putting the above approximations together we obtain that for k and ℓ sufficiently large we have

$$\begin{aligned}&\tilde{c}_1(u,v)^{m_k}\cdot\tilde{c}_2(u\phi(2\ell h_1hm_k,0),v\phi(0,-hm_k))^{n_k}\\&\qquad\qquad\qquad\qquad\cdot\overline{\tilde{c}_1(u,v)^{m_\ell}\cdot\tilde{c}_2(u\phi(2\ell h_1hm_\ell,0),v\phi(0,-hm_\ell))^{n_\ell}}\\&\approx\alpha_1^{m_k-m_\ell}\cdot\big(\text{character}\big)\cdot\alpha_2^{n_k-n_\ell}\cdot\big(\text{character}\big)\\&\quad\cdot\exp\Big(2\pi\mathrm{i}\sum_{j=1}^{J_2}a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}\big(n_k\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\\&\qquad\qquad\qquad\qquad-n_\ell\{\gamma_{2,j}(\phi(2\ell h_1hm_\ell\mathbf{e}_1),\phi(-hm_\ell\mathbf{e}_2))\}\big)\Big),\end{aligned}$$

so defining

$$\begin{aligned}&p(m,n)=\{\alpha_1^m\}+\{\alpha_2^n\}\\&+\sum_{j=1}^{J_2}a_{2,j}\{\chi_{2,j}(\phi(2\ell h_2h\mathbf{e}_1),\phi(-h\mathbf{e}_2))\}n_k\{\gamma_{2,j}(\phi(2\ell h_1hm_k\mathbf{e}_1),\phi(-hm_k\mathbf{e}_2))\}\end{aligned}$$

we see that this is a gen-polynomial not depending on α that has the desired property. $\square$

In Proposition 3.31 we begin to see the makings of the simplification of the expressions

$$\chi_1(\sigma((\ell n^2+an,0),z))\cdot\chi_2(\sigma((\ell n^2+an,n),z)),$$

that was promised immediately after the proof of Lemma 3.26, although it will require some more manipulation before the above proposition bears on this expression directly.

Corollary 3.34. *If $p:\mathbb{Z}^2\to\mathbb{R}$ is the gen-polynomial of Proposition 3.31 then for any $\varepsilon>0$ there are some $K\geq 1$, functions ξ_1, ξ_2, $\ldots$, $\xi_K\in L^2(m_Z)$ and characters $\chi_{i,1},\chi_{i,2},\ldots,\chi_{i,K}\in\widehat{K_1\times K_2}$ for $i=1,2$ such that*

$$\begin{aligned}&\chi_2\circ\sigma((mq_{11}+nq_{21},0),z)\cdot\chi_1\circ\sigma((mq_{11}+nq_{21},mq_{12}+nq_{22}),z)\cdot\overline{\chi_1}\circ\sigma((0,mq_{12}+nq_{22}),z)\\&\qquad\qquad\approx_\varepsilon\exp(-2\pi\mathrm{i}p(m,n))\cdot\sum_{k=1}^{K}\chi_{1,k}(\phi(m\mathbf{q}_1))\chi_{2,k}(\phi(n\mathbf{q}_2))\cdot\xi_k(z)\end{aligned}$$

in $L^2(m_Z)$ for every $m\in\mathbb{Z}$ such that $\phi(m\mathbf{q}_1)\in U_\alpha$.

Proof Letting C_α be the Borel function $Z\times(K_1\times K_2)^2\to\mathrm{S}^1$ output by Proposition 3.31, it will suffice to prove that there are ξ_1, ξ_2, $\ldots$, $\xi_K\in L^2(m_Z)$ and

characters $\chi_{i,1}, \chi_{i,2}, \ldots, \chi_{i,K} \in \widehat{K_1 \times K_2}$ as above such that

$$C_\alpha(\,\cdot\,, u_1, v_1, u_2, v_2) \approx_\varepsilon \sum_{k=1}^{K} \chi_{1,k}(u_1, v_1)\chi_{2,k}(u_2, v_2) \cdot \xi_k \qquad \text{in } L^2(m_Z)$$

for all $(u, v) \in (K_1 \times K_2)^2$.

Proposition 3.31 gives us that the map $(u_1, v_1, u_2, v_2) \mapsto C_\alpha(\,\cdot\,, u_1, v_1, u_2, v_2)$ is continuous from $(K_1 \times K_2)^2$ into $L^2(m_Z)$. This implies that its image is compact, and so lies within the $(\varepsilon/2)$-neighbourhood of some finite-dimensional subspace of $L^2(m_Z)$; let $\xi_1, \xi_2, \ldots, \xi_K$ be a basis for that subspace. Simply by projecting onto this subspace it follows that we can approximate the map $(u_1, v_1, u_2, v_2) \mapsto C_\alpha(\,\cdot\,, u_1, v_1, u_2, v_2)$ uniformly in (u_1, v_1, u_2, v_2) by some map of the form

$$\sum_{m=1}^{M} C_{\alpha,m}(u_1, v_1, u_2, v_2) \cdot \xi_m$$

with each $C_{\alpha,m} : (K_1 \times K_2)^2 \to \mathbb{C}$ a continuous function.

However, now the Stone-Weierstrass Theorem gives for each $C_{\alpha,m}$ a trigonometric polynomial $(K_1 \times K_2)^2 \to \mathbb{C}$ that approximates $C_{\alpha,m}$ uniformly to within $\varepsilon/(2(\|\xi_1\|_2 + \ldots + \|\xi_K\|_2))$. Replacing each c_m by this trigonometric polynomial in our first approximant to C_α and re-arranging the terms gives the result. $\square$

3.6 Completion of the proof

We are finally ready to prove Proposition 3.25.

Proof of Proposition 3.25 By Lemma 3.26 we need only prove convergence of the averages

$$\frac{1}{N}\sum_{n=1}^{N} \theta_1^{\ell n^2 + an}\theta_2^{n} \cdot \chi_1(\sigma((\ell n^2 + an, 0), z)) \cdot \chi_2(\sigma((\ell n^2 + an, n), z)) \cdot f(T_2^n(x))$$

for any $\theta_1, \theta_2 \in \mathrm{S}^1$, and by Lemma 3.27 we may restrict our attention to the case covered by the above results, and in particular Corollary 3.34. We will handle this case in two steps.

Step 1 We first need a simple but slightly fiddly re-arrangement in order to bring Corollary 3.34 to bear, because it applies only to the sublattice $\Gamma = \mathbb{Z}\mathbf{q}_1 +$

$\mathbb{Z}\mathbf{q}_2$ of $\mathbb{Z}^2$. To do this, let us choose an integer $\ell_1 \geq 1$ so that $\ell_1\mathbb{Z}^2 \leq \Gamma$ and break up the above average as

$$\begin{aligned}
&\frac{1}{\ell_1}\sum_{j=1}^{\ell_1}\frac{1}{\lfloor N/\ell_1\rfloor}\sum_{n=0}^{\lfloor N/\ell_1\rfloor}\theta_1^{\ell(\ell_1 n+j)^2+a(\ell_1 n+j)}\theta_2^{\ell_1 n+j}\cdot\chi_1(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),0),z))\\
&\qquad\qquad\cdot\chi_2(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),\ell_1 n+j),z))\cdot f(T_2^{\ell_1 n}(T_2^j(x)))\\
&\qquad\qquad+R\\
&=\frac{1}{\ell_1}\sum_{j=1}^{\ell_1}\theta_1^{\ell j^2+aj}\theta_2^j\frac{1}{\lfloor N/\ell_1\rfloor}\sum_{n=0}^{\lfloor N/\ell_1\rfloor}\theta_1^{\ell_1(\ell\ell_1 n^2+2\ell jn+an)}\theta_2^{\ell_1 n}\cdot\chi_1(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),0),z))\\
&\qquad\qquad\cdot\chi_2(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),\ell_1 n+j),z))\cdot f(T_2^{\ell_1 n}(T_2^j(x)))\\
&\qquad\qquad+R
\end{aligned}$$

where the remainder term satisfies $\|R\|_2 = \mathrm{O}(1/N)$, and so may henceforth be ignored. It will suffice to prove that for each $j \in \{1, 2, \ldots, \ell_1\}$ the inner average over $0 \leq n \leq \lfloor N/\ell_1\rfloor$ converges in $L^2(\mu)$.

To simplify these inner averages, let us recall the consequence of the defining equation for the cocycle σ that we have factorizations

$$\begin{aligned}
&\chi_1(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),0),z))\\
&=\chi_1\big(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z)\big)\cdot\chi_1\big(\sigma((\ell j^2+aj,0),z\cdot\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1))\big)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\chi_2(\sigma((\ell(\ell_1 n+j)^2+a(\ell_1 n+j),(\ell_1 n+j)),z))\\
&=\chi_2\big(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),\ell_1 n),z)\big)\\
&\qquad\cdot\chi_2\big(\sigma((\ell j^2+aj,j),z\cdot\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1+\ell_1 n\mathbf{e}_2))\big).
\end{aligned}$$

Now, for fixed integers ℓ_1 and j the second factors in the factorizations above correspond to the functions

$$h_1 : z \mapsto \chi_1(\sigma((\ell j^2+aj,0),z))$$

and

$$h_2 : z \mapsto \chi_2(\sigma((\ell j^2+aj,j),z)),$$

so that we can write

$$\begin{aligned}
&\chi_1\big(\sigma((\ell j^2+aj,0),z\cdot\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1))\big)\\
&\qquad\cdot\chi_2\big(\sigma((\ell j^2+aj,j),z\cdot\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1+\ell_1 n\mathbf{e}_2))\big)
\end{aligned}$$

as

$$h_1(R_{\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1)}z)\cdot h_2(R_{\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1+\ell_1 n\mathbf{e}_2)}z).$$

Since we may approximate each of h_1 and h_2 arbitrarily well in $L^2(m_Z)$ by a trigonometric polynomial on Z, it follows by continuity and multilinearity that the desired convergence will follow if we prove it instead for the averages

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N}\theta_1^{\ell_1(\ell\ell_1 n^2+2\ell jn+an)}\theta_2^{\ell_1 n}\cdot\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z))\\ \cdot\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),\ell_1 n),z))\\ \cdot h_1(R_{\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1)}z)\cdot h_2(R_{\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1+\ell_1 n\mathbf{e}_2)}z)\\ \cdot f(T_2^{\ell_1 n}(T_2^j(x)))\end{aligned}$$

where each of h_1 and h_2 is a character. In that case

$$h_1(R_{\phi(\ell_1(\ell\ell_1 n^2+2\ell jn+an)\mathbf{e}_1)}z)=h_1(\phi(\mathbf{e}_1))^{\ell_1(\ell\ell_1 n^2+2\ell jn+an)}h_1(z)$$

and similarly for h_2, so by taking the n-independent functions $h_1(z)$ and $h_2(z)$ outside the average and adjusting the values of θ_1 and θ_2 we can now drop the mention of these functions h_i altogether to leave the averages

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\theta_1^{\ell_1(\ell\ell_1 n^2+2\ell jn+an)}\theta_2^{\ell_1 n}\cdot\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z))\\ \cdot\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),\ell_1 n),z))\cdot f(T_2^{\ell_1 n}(T_2^j(x))).\end{aligned}$$

Step 2 The value of the simplification achieved in Step 1 above is that now by our choice of ℓ_1 we have $(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n)\in\Gamma$ for all $n\geq 1$. In particular, it follows that there are independent linear forms $L_1,L_2:\Gamma\to\mathbb{Z}$ such that

$$\begin{aligned}(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n)=L_1(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n)\mathbf{q}_1\\ +L_2(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n)\mathbf{q}_2\end{aligned}$$

for all n. Let us abbreviate $\vec{L}:=(L_1,L_2)$ and

$$Q_i(n):=L_i(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n),$$

so that Q_1 and Q_2 are two non-constant, linearly independent quadratic functions $\mathbb{Z}\to\mathbb{Z}$.

Now recall the open subsets $U_\alpha \subseteq K_1 \times K_2$ introduced in Proposition 3.31. The set

$$\{m \in \mathbb{Z} : \ \phi(m\mathbf{q}_1) \in U_\alpha\}$$

is a Bohr set in $\mathbb{Z}$, and by construction it is defined by irrational phases. Consequently, the multidimensional version of Weyl's Equidistribution Theorem (see, for instance, Theorem 1.6.4 in Kuipers and Niederreiter [19]) gives that the set

$$E_\alpha := \{n \geq 1 : \ \phi(Q_1(n)\mathbf{q}_1) \in U_\alpha\}$$

has asymptotic density equal to $m_{K_1\times K_2}(U_\alpha) > 1-\alpha$. Since the terms of our average

$$\frac{1}{N}\sum_{n=1}^{N}\theta_1^{\ell_1(\ell\ell_1 n^2+2\ell jn+an)}\theta_2^{\ell_1 n}\cdot\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell kn+an),0),z)) \\ \cdot\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),\ell_1 n),z))\cdot f(T_2^{\ell_1 n}(T_2^{j}(x))).$$

are uniformly bounded in L^∞, to prove norm convergence it suffices to prove it for the related averages in which we restrict the sum to those n that lie inside some subset of $\mathbb{N}$, provided we can choose that set to have arbitrarily high asymptotic density. Hence, in particular, it will suffice to prove for every $\alpha > 0$ the convergence of the averages in which we restrict the summation to $n \in \{1,2,\dots,N\}\cap E_\alpha$.

Now, Corollary 3.34 gives a gen-polynomial $p : \mathbb{Z}^2 \to \mathbb{R}$, and for any $\alpha > 0$ and $\varepsilon > 0$ some functions ξ_1, ξ_2, $\dots$, $\xi_K \in L^2(m_Z)$ and characters $\chi_{i,1}$, $\chi_{i,2}$, $\dots$, $\chi_{i,K} \in \widehat{K_1\times K_2}$ for $i = 1,2$ such that

$$\begin{aligned}&\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z))\cdot\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n),z)) \\ &\qquad\qquad\qquad\cdot\overline{\chi_1(\sigma((0,-\ell_1 n),z))} \\ &\approx_\varepsilon \exp(-2\pi\mathrm{i}\,p\circ\vec{L}(\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n)) \\ &\qquad\qquad\qquad\cdot\sum_{k=1}^{K}\chi_{1,k}(\phi(Q_1(n)\mathbf{q}_1))\chi_{2,k}(\phi(Q_2(n)\mathbf{q}_2))\cdot\xi_k(z)\end{aligned}$$

in $L^2(m_Z)$ for all $n \geq 1$ with $n \in E_\alpha$. Using the cocycle equation we can re-write

$$\begin{aligned}&\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z))\cdot\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),\ell_1 n),z)) \\ &=\chi_1(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),-\ell_1 n),z\phi(0,\ell_1 n)))\cdot\chi_1(\sigma((0,\ell_1 n),z)) \\ &\qquad\cdot\chi_2(\sigma((\ell_1(\ell\ell_1 n^2+2\ell jn+an),0),z\phi(0,\ell_1 n)))\cdot\chi_2(\sigma((0,\ell_1 n),z)),\end{aligned}$$

and now substituting from the above approximation we see that for all $n \in \mathbb{N} \cap E_\alpha$ this lies within ε in $L^2(m_Z)$ of

$$\begin{aligned}
&\exp(-2\pi\mathrm{i}\, p \circ \vec{L}(\ell_1(\ell\ell_1 n^2 + 2\ell j n + an), -\ell_1 n)) \\
&\qquad\qquad \cdot \sum_{k=1}^{K} \chi_{1,k}(\phi(Q_1(n)\mathbf{q}_1))\chi_{2,k}(\phi(Q_2(n)\mathbf{q}_2)) \cdot \xi_k(z\phi(0,\ell_1 n)) \\
&\qquad\qquad \cdot \chi_2(\sigma((0,\ell_1 n), z)) \cdot \chi_1(\sigma((0,\ell_1 n), z)) \cdot \chi_1(\sigma((0,-\ell_1 n), z\phi(0,\ell_1 n))) \\
&= \exp(-2\pi\mathrm{i}\, p \circ \vec{L}(\ell_1(\ell\ell_1 n^2 + 2\ell j n + an), -\ell_1 n)) \\
&\qquad\qquad \cdot \sum_{k=1}^{K} \chi_{1,k}(\phi(Q_1(n)\mathbf{q}_1))\chi_{2,k}(\phi(Q_2(n)\mathbf{q}_2)) \cdot \xi_k(z\phi(0,\ell_1 n))\chi_2(\sigma((0,\ell_1 n), z)),
\end{aligned}$$

using that the cocycle equation also gives

$$\sigma((0,\ell_1 n), z) \cdot \sigma((0,-\ell_1 n), z\phi(0,\ell_1 n)) = \sigma((0,0), z) = 1.$$

Since $\varepsilon > 0$ was arbitrary we may substitute this approximation into our averages above and appeal again to multilinearity to deduce that it suffices to prove instead the norm convergence of the averages

$$\begin{aligned}
\frac{1}{N} \sum_{1 \le n \le N,\, n \in E_\alpha} &\theta_1^{\ell_1(\ell\ell_1 n^2 + 2\ell j n + an)} \theta_2^{\ell_1 n} \cdot \exp(-2\pi\mathrm{i}\, p \circ \vec{L}(\ell_1(\ell\ell_1 n^2 + 2\ell j n + an), -\ell_1 n)) \\
&\qquad\qquad \cdot \tilde{\chi}_1(\phi(Q_1(n)\mathbf{q}_1))\tilde{\chi}_2(\phi(Q_2(n)\mathbf{q}_2)) \\
&\qquad\qquad \cdot \xi(z\phi(0,\ell_1 n)) \cdot \chi_2(\sigma((0,\ell_1 n), z)) \cdot f(T_2^{\ell_1 n}(T_2^k(x)))
\end{aligned}$$

for any two characters $\tilde{\chi}_1, \tilde{\chi}_2 \in \widehat{K_1 \times K_2}$ and any fixed function $\xi \in L^2(m_Z)$.

Finally, in order to prove convergence we may freely insert the n-independent function $(z, a) \mapsto \chi_2(a)$ into these averages, because this function is bounded away from zero. This trick now leads to the simplification

$$\xi(z\phi(0,\ell_1 n)) \cdot \chi_2(a) \cdot \chi_2(\sigma((0,\ell_1 n), z)) \cdot f(T_2^{\ell_1 n}(T_2^k(x))) = F(T_2^{\ell_1 n}(x))$$

where $F(x) := \xi(z) f(T_2^k(x)) \chi_2(a)$ (remembering that $(z, a) = \eta(x)$). On the other hand, the expression

$$\begin{aligned}
\theta_1^{\ell_1(\ell\ell_1 n^2 + 2\ell j n + an)} \theta_2^{\ell_1 n} &\cdot \exp(-2\pi\mathrm{i}\, p \circ \vec{L}(\ell_1(\ell\ell_1 n^2 + 2\ell j n + an), -\ell_1 n)) \\
&\qquad\qquad \cdot \tilde{\chi}_1(\phi(Q_1(n)\mathbf{q}_1))\tilde{\chi}_2(\phi(Q_2(n)\mathbf{q}_2))
\end{aligned}$$

clearly just defines an expression of the form $\exp(\mathrm{i}Q_3(n))$ for $Q_3 : \mathbb{Z} \to \mathbb{R}$ a new gen-polynomial, and so the rather unwieldy averages above can be written in the simple form

$$\frac{1}{N}\sum_{1\leq n\leq N,\, n\in E_\alpha}\exp(\mathrm{i}Q_3(n))\cdot F\circ T_2^{\ell_1 n} = \frac{1}{N}\sum_{n=1}^{N}1_{E_\alpha}(n)\cdot\exp(\mathrm{i}Q_3(n))\cdot F\circ T_2^{\ell_1 n}.$$

Next, the indicator function 1_{E_α} corresponds to a quadratic Bohr set, and so among 1-bounded functions on $\mathbb{N}$ it can be approximated in density by linear combinations of gen-polynomial maps taking values in S^1. Appealing once again to multilinearity, it follows that we need only prove convergence of the averages

$$\frac{1}{N}\sum_{n=1}^{N}\exp(\mathrm{i}Q_4(n))\cdot F\circ T_2^{\ell_1 n}$$

for a suitably-enlarged list of possible gen-polynomials Q_4.

The convergence of these now follows from the results of Bergelson and Leibman in [8] (or could probably also be deduced from the results of Host and Kra in their related paper [18]). In particular, a simple appeal to the spectral theorem and Corollary 0.26 in [8] shows that whenever $(U_1^t)_{t\in\mathbb{R}}$ and U_2 are respectively a unitary flow and a unitary operator acting on a Hilbert space $\mathfrak{H}$ and $Q_1' : \mathbb{Z}\to\mathbb{R}$ and $Q_2' : \mathbb{Z}\to\mathbb{R}$ are generalized polynomials, then the sequence of operator averages

$$\frac{1}{N}\sum_{n=1}^{N}U_1^{Q_1'(n)}U_2^{Q_2'(n)}$$

converges in the strong operator topology. (In fact this result lies just between two further corollaries that Bergelson and Leibman obtain explicitly in [8], Corollary 0.27 concerning tuples of flows and Corollary 0.28 concerning tuples of single operators.) This implies the convergence we need in the case when $\mathfrak{H} = L^2(\mu)$, U_1^t is multiplication by $\exp(\mathrm{i}t)$, U_2 is the Koopman operator of $T_2^{\ell_1}$, $Q_1'(n) := Q_4(n)$ and $Q_2'(n) := n$.

This completes the proof of Proposition 3.25, and hence of Theorem 1.1. $\square$

Remark In [16] Host and Kra augment their proof of convergence with a description of the limit function that emerges. Although the last step in our proof of convergence above is rather similar to their argument, the other stages in our reduction leave it much less clear just how the limit function can be described in our case, even after passing to a suitable extended system. $\triangleleft$

A Moore cohomology

We collect here the definition of Moore's measurable cohomology theory for locally compact groups and some of its basic properties that are needed in Section 2. Some of the result proved below can be improved using the continuity results of [1], but we have left them in the form in which they were presented before the appearance of that paper in order to remain consistent with the main text above.

The most convenient definition of this cohomology theory for our purposes is in terms of the measurable homogeneous bar resolution. We recall this here for completeness, noting that it is shown by Moore to be equivalent to various more abstract definitions, and to support the usual functorial cohomological machinery of discrete group cohomology (particularly the procedure of dimension-shifting and the Hochschild-Serre spectral sequence).

Definition A.1 (Measurable cohomology for locally compact groups)**.** *If A is a locally compact group, R is a Polish Abelian group and $\alpha : A \curvearrowright R$ is a continuous left-action by automorphisms, then we define the* ***measurable cohomology of*** A ***with coefficients in*** (R,α) *as the (discrete) cohomology of the chain complex*

$$0 \longrightarrow R \xrightarrow{d} \mathcal{C}(A,R) \xrightarrow{d} \mathcal{C}(A^2,R) \xrightarrow{d} \dots$$

with chain maps defined by

$$d\phi(a_1,a_2,\dots,a_{n+1}) := \alpha^{a_{n+1}}(\phi(a_1,a_2,\dots,a_n))$$
$$+\sum_{i=1}^{n}(-1)^{n+1-i}\phi(a_1,a_2,\dots,a_i+a_{i+1},\dots,a_{n+1})+(-1)^{n+1}\phi(a_2,a_3,\dots,a_{n+1}).$$

We write $\mathcal{Z}^n(A,R) := \ker d|_{\mathcal{C}(A^n,R)}$ for the subgroup of ***cocycles*** *in $\mathcal{C}(A^n,R)$ and $\mathcal{B}^n(A,R) := \mathrm{img}\, d|_{\mathcal{C}(A^{n-1},R)}$ for the subgroup of* ***coboundaries****, and in these terms the cohomology groups are the* discrete *groups*

$$\mathrm{H}^n(A,R) := \frac{\mathcal{Z}^n(A,R)}{\mathcal{B}^n(A,R)}.$$

We warn the reader that this definition of differential is 'back-to-front' compared with the usual conventions of discrete group cohomology (see Section 6.5 of Weibel [26]) so as to be better adapted to our present setting; it is clear that this makes only a cosmetic difference to the theory.

It is easy to find examples in which the measurability condition on the above cochains makes a large difference to the cohomology groups that result. Perhaps

most simply, it is easy to check that for any Polish Abelian group A with trivial $\mathbb{R}$-action we have that $\mathrm{H}^1(\mathbb{R}, A)$ is isomorphic to the group of continuous homomorphisms $\mathbb{R} \to R$, whereas $\mathrm{H}^1(\mathbb{R}_{\mathrm{discrete}}, A)$ is a discrete Abelian group of uncountable rank in general.

Moore also gives some discussion in [24] of possible topologies on the cohomology groups themselves. However, the obvious candidate topologies are often badly behaved (for example, by being non-Hausdorff, as in the well-known case when $A = \mathbb{Z}$, $R = \mathcal{C}(X, \mu)$ and $\alpha^n(f) = f \circ T^n$ for some nontrivial aperiodic action $T : \mathbb{Z} \curvearrowright (X, \mu)$), and we will not need a topology on these groups here.

We now state three important calculational results from Moore's papers that we will need later. Their proofs employ the basic functorial machinery of this cohomology theory that are set up there, particularly the Hochschild-Serre spectral sequence and its corollary, the restriction-inflation exact sequence; we omit them here.

Proposition A.2 (Second cohomology and the fundamental group)**.** *If Z is a compact connected Lie group with fundamental group $\pi_1(Z)$, and $\pi_1(Z)_{\mathrm{tor}}$ is the torsion subgroup of $\pi_1(Z)$, then there is a canonical isomorphism $\mathrm{H}^2(Z, \mathbb{T}) \cong \widehat{\pi_1(Z)_{\mathrm{tor}}}$. In particular, $\mathrm{H}^2(\mathbb{T}^d, \mathbb{T}) = 0$ for all $d \geq 1$.*

Proof This is Proposition 2.1 in part I of [22]. ☐

Proposition A.3 (Continuity of H^2 under inverse and direct limits)**.** *If $Z = \lim_{m\leftarrow} Z_{(m)}$ is an inverse limit of compact groups and $A = \lim_{m\rightarrow} A_{(m)}$ is a direct limit of countable discrete groups with trivial Z-action then*

1. *$\mathrm{H}^2(Z, A)$ is isomorphic to the direct limit of the groups $\mathrm{H}^2(Z_{(m)}, A_{(m)})$ under the compositions of the inflation maps $\inf : \mathrm{H}^2(Z_{(m)}, A_{(m)}) \hookrightarrow \mathrm{H}^2(Z, A_{(m)})$ with the embeddings $A_{(m)} \to A$, and*

2. *$\mathrm{H}^2(Z, \mathbb{T})$ is similarly isomorphic to the direct limit of the groups $\mathrm{H}^2(Z_{(m)}, \mathbb{T})$ under the inflation maps $\inf : \mathrm{H}^2(Z_{(m)}, \mathbb{T}) \to \mathrm{H}^2(Z, \mathbb{T})$.*

Proof These are special cases of Theorems 2.1 and 2.2 of Part I of [22] (observing that any compact Abelian group is almost connected). ☐

Lemma A.4 (Real cohomology of compact Abelian groups)**.** *If Z is a compact Abelian group then $\mathrm{H}^1(Z, \mathbb{R}) = \mathrm{H}^2(Z, \mathbb{R}) = 0$. If Z is a finite-dimensional compact Abelian group then this extends to $\mathrm{H}^n(Z, \mathbb{R}) = 0$ for all $n > 0$.*

Proof The first conclusion is part of Theorem 2.3 in Part I of Moore [22], and the second follows from the identification for compact Lie groups of Moore's measurable cohomology with the cohomology theory for topological groups defined using classifying spaces, as outlined by Moore at the end of [23] and described in detail by Wigner in [27]. $\square$

Lemma A.5 (Integral degree-2 cohomology)**.** *If Z is a compact Abelian group then $\mathrm{H}^2(Z,\mathbb{Z}) \cong \widehat{Z}$, where the isomorphism is given by assigning to $\gamma \in \widehat{Z}$ the 2-cocycle*

$$\kappa_\gamma(z,w) := \lfloor \{\gamma(z)\} + \{\gamma(w)\} \rfloor.$$

Proof Suppose that $\kappa : Z \times Z \to \mathbb{Z} \subset \mathbb{R}$ is a Borel 2-cocycle. By the previous lemma we know there is some $a : Z \to \mathbb{R}$ such that $da = \kappa$, but of course this a may not be $\mathbb{Z}$-valued. However, since κ does take values in $\mathbb{Z}$, we know that

$$a(z) + a(w) - a(z+w) + \mathbb{Z} = \kappa(z,w) + \mathbb{Z} = \mathbb{Z}$$

almost surely, so on composing with the quotient map $\mathbb{R} \to \mathbb{T}$ our 1-cochain a must descend to a measurable (and hence continuous) character $\gamma \in \widehat{Z}$. The map $a'(z) := \{\gamma(z)\} \in [0,1)$ clearly does give γ upon composing with the quotient, and on the other hand a direct computation gives

$$a'(z) + a'(w) - a'(z+w) = \kappa_\gamma(z,w)$$

(since $a + b - \{a+b\} \equiv \lfloor a+b \rfloor$ for $a,b \in [0,1)$). Therefore $\kappa - \kappa_\gamma = d(a - a')$ with $a - a'$ taking values in $\mathbb{Z}$.

On the other hand any two 2-cocycles of the form κ_γ must give rise to different homomorphisms above, and so they cannot be cohomologous in $\mathcal{Z}^2(Z,\mathbb{Z})$. This completes the proof. $\square$

Remark In fact for $Z = \mathbb{T}^d$ the preceding lemma is a special case of a rather more far-reaching description of the integral cohomology. With the standard definition of cup product, the cohomology ring $\mathrm{H}^*(\mathbb{T}^d,\mathbb{Z})$ is isomorphic to the polynomial ring $\mathbb{Z}[X_1, X_2, \ldots, X_d]$ graded so that each free variable X_i has degree two (so, in particular, $\mathrm{H}^n(\mathbb{T}^d,\mathbb{Z}) = 0$ when n is odd), and for even n the cochains

$$\begin{aligned} &c(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n) \\ &\qquad := \Big(\prod_{j=1}^{d} \prod_{i=1}^{\ell_j} \lfloor \{t_{2i-1, 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{j-1} + j}\} + \{t_{2i, 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{j-1} + j}\} \rfloor \Big) \end{aligned}$$

corresponding to the monomials $X_1^{\ell_1}X_2^{\ell_2}\cdots X_d^{\ell_d}$ with $2\ell_1+2\ell_2+\cdots+2\ell_d=n$ comprise a free set of generators of $\mathrm{H}^n(\mathbb{T}^d,\mathbb{Z})$, where we write $\mathbf{t}_i=(t_{i,1},t_{i,2},\ldots,t_{i,d})\in\mathbb{T}^d$. In all cases these calculations can be performed directly using the measurable versions of standard group cohomological machinery, particularly the Hochschild-Serre spectral sequence, that are set up in Moore's earlier papers [22]; or, alternatively, they can be deduced from results of Wigner [27] showing that for $\mathbb{T}^d$ and these particular target modules the Moore cohomology can be identified with various other cohomology theories (such as that defined in terms of classifying spaces, developed in detail for compact Abelian groups by Hofmann and Mostert in [15]). $\lhd$

The proof of Proposition 2.1 in Section 2 will rest on the following rather more detailed cohomological calculations.

Lemma A.6. *Suppose that F is a finite Abelian group, $r\geq 0$, G is another locally compact Abelian group on which $\mathbb{T}^r\times F$ acts trivially,*

$$\kappa:(\mathbb{T}^r\times F)\times(\mathbb{T}^r\times F)\to G$$

is a 2-cocycle and

$$\beta:(\mathbb{T}^r\times F)^3\to G$$

is a 3-cocycle.

Then

1. *if $G=\mathbb{T}$ then κ is cohomologous to a 2-cocycle κ' that depends only on the coordinates in F;*
2. *if $G=\mathbb{Z}$ then β is cohomologous to a 3-cocycle β' that depends only on the coordinates in F;*
3. *if $G=\mathbb{T}$ and κ is a $\mathbb{T}$-valued coboundary on $\mathbb{T}^r\times F$ and depends only on coordinates in F, then κ is is a $\mathbb{T}$-valued coboundary on F;*
4. *if $G=\mathbb{Z}/n\mathbb{Z}$ then κ is cohomologous to a 2-cocycle κ' of the form $\kappa'(z,w):=\kappa''(z,w)+\lfloor\{\gamma(z)\}+\{\gamma(w)\}\rfloor+n\mathbb{Z}$ for some $\gamma\in\widehat{\mathbb{T}^r\times F}$ and some 2-cocycle κ'' that depends only on coordinates in F.*

Proof 1. The first conclusion follows from the spectral sequence calculations of Section 3 in Part I of Moore [22]. In particular, the first two layers of the Hochschild-Serre spectral sequence introduce a filtering of groups

$$\mathrm{H}^2(\mathbb{T}^r\times F,\mathbb{T})\geq K_1\geq K_2\geq\{0\}$$

where K_1 is identified with the subgroup of cohomology classes containing a representative 2-cocycle κ such that $\kappa|_{\mathbb{T}^r\times\mathbb{T}^r} = 0$ (that is, the kernel of the restriction map to $\mathbb{T}^r$), K_2 with the further subgroup of classes containing a representative that depends only coordinates in F (that is, the image of the inflation map), and such that $K_2/K_1 \cong \mathrm{H}^1(F, \mathrm{H}^1(\mathbb{T}^r, \mathbb{T}))$ (where $\mathrm{H}^1(\mathbb{T}^r, \mathbb{T})$ is given the discrete topology).

However, Proposition A.2 tells us that $\mathrm{H}^2(\mathbb{T}^r, \mathbb{T}) = 0$, so for any 2-cocycle $\kappa : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to \mathbb{T}$ we can find some $\alpha : \mathbb{T}^r \to \mathbb{T}$ such that $\kappa|_{\mathbb{T}^r\times\mathbb{T}^r} = d\alpha$. If we lift α to $\mathbb{T}^r \times F$ under the coordinate projection map, it follows that $\kappa - d\alpha$ is a cohomologous 2-cocycle that vanishes on $\mathbb{T}^r \times \mathbb{T}^r$, and so we have shown that in our setting $\mathrm{H}^2(\mathbb{T}^r \times F, \mathbb{T}) = K_1$.

In addition, we know that $\mathrm{H}^1(\mathbb{T}^r, \mathbb{T}) = \widehat{\mathbb{T}^r} \cong \mathbb{Z}^r$ is torsion-free, and so $\mathrm{H}^1(F, \mathrm{H}^1(\mathbb{T}^r, \mathbb{T})) \cong \mathrm{Hom}(F, \mathbb{Z}^r) = 0$. Thus in fact $\mathrm{H}^2(\mathbb{T}^r \times F, \mathbb{T}) = K_2$, giving the first conclusion is proved.

2. This will follow from Part 1 and the switchback maps of the long exact sequence

$$\begin{aligned}\ldots \to \mathrm{H}^n(\mathbb{T}^r \times F, \mathbb{Z}) \to \mathrm{H}^n(\mathbb{T}^r \times F, \mathbb{R}) \to \mathrm{H}^n(\mathbb{T}^r \times F, \mathbb{T})& \\ \xrightarrow{\text{switchback}} \mathrm{H}^{n+1}(\mathbb{T}^r \times F, \mathbb{Z}) \to \mathrm{H}^{n+1}(\mathbb{T}^r \times F, \mathbb{R}) \to \ldots&\end{aligned}$$

corresponding to the presentation $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{T}$. By Lemma A.4 we have $\mathrm{H}^n(\mathbb{T}^r \times F, \mathbb{R}) = 0$ for all $n \geq 1$, so this long exact sequence collapses to a collection of isomorphisms

$$\mathrm{H}^n(\mathbb{T}^r \times F, \mathbb{T}) \cong \mathrm{H}^{n+1}(\mathbb{T}^r \times F, \mathbb{Z})$$

which for $n = 2$ directly enables us to appeal to Part 1.

More explicitly, given any 3-cocycle $\beta : (\mathbb{T}^r \times F)^3 \to \mathbb{Z}$, we can express it as the coboundary of an $\mathbb{R}$-valued 2-cochain $\kappa : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to \mathbb{R}$, and now since β takes values in $\mathbb{Z}$ it follows that $\kappa + \mathbb{Z}$ is a $\mathbb{T}$-valued 2-cocycle. Therefore by Part 1 we can find some $\alpha_0 : \mathbb{T}^r \times F \to \mathbb{T}$ such that $\kappa_0' := (\kappa + \mathbb{Z}) - d\alpha_0$ depends only on coordinates in F. Now let $\alpha : \mathbb{T}^r \times F \to \mathbb{R}$ be a lift of α_0 and $\kappa' : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to \mathbb{R}$ a lift of κ_0' that depends only on coordinates in F, so we must have that $\kappa'' := \kappa - d\alpha - \kappa'$ is $\mathbb{Z}$-valued. It follows that $\beta = d\kappa = d\kappa' + d\kappa''$, where κ' depends only on coordinates in F and κ'' is $\mathbb{Z}$-valued, as required.

3. We need to show that the inflation map $\inf : \mathrm{H}^2(F, \mathbb{T}) \to \mathrm{H}^2(\mathbb{T}^r \times F, \mathbb{T})$ is injective. This follows from another consequence of Moore's spectral sequence calculations: the measurable analog of Lyndon's inflation-restriction exact sequence,

derived in Section I.5 of Part I of [22]. In our case this specializes to

$$0 \to \mathrm{H}^1(F,\mathbb{T}) \xrightarrow{\mathrm{inf}} \mathrm{H}^1(\mathbb{T}^r \times F,\mathbb{T}) \xrightarrow{\mathrm{res}} \mathrm{H}^1(\mathbb{T}^r,\mathbb{T})$$
$$\xrightarrow{\mathrm{tg}} \mathrm{H}^2(F,\mathbb{T}) \xrightarrow{\mathrm{inf}} \mathrm{inf}(\mathrm{H}^2(F,\mathbb{T})) \leq \mathrm{H}^2(\mathbb{T}^r \times F,\mathbb{T}),$$

where tg is the so-called 'transgression' map. We do not need the precise definition of tg, but only the result of Moore that it is zero for a split extension such as $\mathbb{T}^r \times F \twoheadrightarrow F$, so that the desired injectivity follows.

4. In view of the presentation

$$\mathbb{Z} \hookrightarrow n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$$

any 2-cocycle $\kappa : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to \mathbb{Z}/n\mathbb{Z}$ lifts to a 2-cochain $\kappa' : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to \mathbb{Z}$, whose coboundary now defines a 3-cocycle $d\kappa' : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to n\mathbb{Z}$. By Part 2 this is cohomologous as an $n\mathbb{Z}$-valued 3-cocycle to some cocycle depending only on the coordinates in F: that is, there are a 2-cochain $\alpha : (\mathbb{T}^r \times F) \times (\mathbb{T}^r \times F) \to n\mathbb{Z}$ and a 3-cocycle $\beta : F \times F \times F \to n\mathbb{Z}$ such that $d\kappa' = d\alpha + \beta$.

Therefore $\beta = d(\kappa' - \alpha)$ is a 3-cocycle depending only on coordinates in F that can be expressed as the coboundary of some $\mathbb{Z}$-valued 2-cochain on $\mathbb{T}^r \times F$, say $\xi_1 \in \mathcal{C}((\mathbb{T}^r \times F)^2, \mathbb{Z})$. We will next show that ξ_1 can also be taken to depend only on coordinates in F.

Using once again the presentation $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{T}$ and Lemma A.4 we see that β can alternatively be expressed as the coboundary of some $\mathbb{R}$-valued 2-cochain on F, say $\xi_2 \in \mathcal{C}(F^2, \mathbb{R})$. Now $d(\xi_2 - \xi_1) = 0$, so $\xi_2 - \xi_1$ is an $\mathbb{R}$-valued 2-cocycle on $\mathbb{T}^r \times F$, so another appeal to the vanishing of real-valued cohomology allows us to write it as $d\gamma_1$ for some Borel $\gamma_1 : \mathbb{T}^r \times F \to \mathbb{R}$. Recalling that ξ_1 is $\mathbb{Z}$-valued, composing with the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ we deduce that $d(\gamma_1 + \mathbb{Z}) = \xi_2 + \mathbb{Z}$. Therefore the $\mathbb{T}$-valued 2-cocycle $\xi_2 + \mathbb{Z}$ on F is a coboundary when lifted to $\mathbb{T}^r \times F$, and so by Part 3 above it is actually a coboundary among cochains that depend only on F. Letting γ_2 be a cochain $F \to \mathbb{R}$ such that $d(\gamma_2 + \mathbb{Z}) = \xi_2 + \mathbb{Z}$, it follows that we have $\beta = d\xi_2 = d(\xi_2 - d\gamma_2)$ where $\xi_2 - d\gamma_2$ takes values in $\mathbb{Z}$. Thus we have shown that β is actually a 3-coboundary for $\mathbb{Z}$-valued cochains depending only on coordinates in F, and hence we can write $\beta = d\kappa''$ for some $\kappa'' : F \times F \to \mathbb{Z}$.

Therefore $d(\kappa' - \alpha - \kappa'') = 0$, so now $\kappa' - \alpha - \kappa''$ is a $\mathbb{Z}$-valued 2-cocycle on $\mathbb{T}^r \times F$, and hence by Lemma A.5 there are some $\gamma \in \widehat{\mathbb{T}^r \times F}$ and cochain $\rho : \mathbb{T}^r \times F \to \mathbb{Z}$

such that

$$(\kappa' - \alpha - \kappa'')(z, w) = d\rho(z, w) + \lfloor \{\gamma(z)\} + \{\gamma(w)\} \rfloor,$$

and so finally since α takes values in $n\mathbb{Z}$, passing back down through the quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ we obtain

$$\kappa(z, w) = (\kappa'' + n\mathbb{Z})(z, w) + d(\rho + n\mathbb{Z})(z, w) + (\lfloor \{\gamma(z)\} + \{\gamma(w)\} \rfloor + n\mathbb{Z}).$$

Since κ'' depends only on coordinates in F this is of the form desired. $\square$

Remark For Part 2 above we made use of the injectivity of certain inflation maps from $\mathrm{H}^*(F, \cdot)$ to $\mathrm{H}^*(F \times H, \cdot)$ for a direct product group $F \times H$. In the setting of finite groups F and H this simple result can be proved by hand using the homogeneous bar resolution. However, in the setting of measurable cohomology on non-finite groups this approach runs into trouble because it relies on sampling cochains on zero-measure subsets of the product group, and our cochains are only defined up to negligible sets. For this reason rigorous proofs require some more careful machinery (particularly the Hochschild-Serre spectral sequence), and take rather more work. $\lhd$

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